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Root Multiplicity for Some GKM Algebras of Order 4

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ABSTRACT
In this paper, we consider the generalized Kac-Moody algebras associated with Generalized Generalized Cartan matrices of finite A_4 family. The root multiplicity of all roots is determined using Witt partition function.
KEYWORDS – *Cartan matrix, Dimension, Hyperbolic, Partition function, Root Multiplicities.*

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I. INTRODUCTION

Borcherds introduced the notion of generalized Kac-Moody algebras (GKM algebras) in [2]. Determining the multiplicities for imaginary roots is still an open problem. Kim and Shin, computed the recursion dimension formula for all graded Lie algebras in [5]. A closed form root multiplicity formula for all the roots of GKM algebras were derived by Kang in [6] and [7]. Root multiplicities of the indefinite kac-Moody algebras $HD_4^{(3)}$, $HG_2^{(1)}$ and $HD_n^{(1)}$ were determined in [3] and [4]. The classification of purely imaginary, Strictly imaginary and special imaginary roots were given in [8] - [12]. The EB_2 family was studied in [13] and [14]. In this paper, we consider the finite A_4 family and compute the root multiplicities for the Generalized Kac-Moody algebras.

II. PRELIMINARIES

The definitions and notations are used as in [1], [15] and [16]. Let I={1,2,3,…} be a finite or countably infinite index set and $A = (a_{i,j})_{i,j)} \in I$ be a real matrix satisfying the following conditions:

1. either $a_{ii} = 2$ or $a_{ii} \leq 0 \forall i \in I$;
2. $a_{ij} \leq 0$ if $i \neq j$ and $a_{ij} \in \mathbb{Z}$ if $a_{ij} = 2$;
3. $a_{ij} = 0$ implies $a_{ij} = 0$.

A is called a generalized generalized Cartan matrix (abbreviated as GGCM) and the Lie algebra $g(A)$ associated with A is called the generalized Kac-Moody algebra. We assume that a GGCM A is symmetrizable if $\exists$ a diagonal matrix $D = diag(s_i | i \in I)$ with $s_i > 0 (i \in I)$ such that DA is symmetric.

Let $I^{re} = \{i \in I | a_{ii} = 2\}$, $I^{im} = \{i \in I | a_{ii} \leq 0\}$, and let $\underline{m} = (m_i \in \mathbb{Z}_{>0} | i \in I)$ be a collectioon of positive integers such that $m_i = 1$ for all $i \in I^{re}$.

The GKM algebra $g = g(A, \underline{m})$ associated with a symmetrizable GGCM $A = (a_{ij})_{i,j \in I}$ of charge $\underline{m} = (m_i | i \in I)$ is the Lie algebra generated by the elements $h_i, d_i (i \in I), e_{ik}, f_{ik} (i \in I, k = 1 \cdots, m_i)$ with the defining relations:

$[h_i, h_j] = [d_i, d_j] = [h_i, d_j] = 0,$
$[h_i, e_{jl}] = a_{ij} e_{jl}, [h_i, f_{jl}] = -a_{ij} f_{jl},$
$[d_i, e_{jl}] = \delta_{ij} e_{jl}, [d_i, f_{jl}] = -\delta_{ij} f_{jl},$
$[e_{ik}, f_{jl}] = \delta_{ij} \delta_{kl} h_i, (ad e_{ik})^{1-a_{ij}}(e_{jl}) = (ad f_{ik})^{1-a_{ij}}(f_{jl}) = 0$ if $a_{ii} = 2, i \neq j,$
$[e_{ik}, e_{jl}] = [f_i k, f_{jl} = 0$ if $a_{ij} = 0$
where, $(i, j \in I, k = 1, \cdots, m_i, l = 1, \cdots, m_j)$.

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Let $P^+ = \{\lambda \in h^* | \lambda(h_i) \geq 0$ for all $i \in I$, $\lambda(h_i) \in \mathbb{Z}_{\geq} 0$ if $a_{ii} = 2\}$, and let $V(\lambda)$ be the irreducible highest weight module over g with highest weight λ.

Let $W_J = \langle r_j | j \in J \rangle$ be the subgroup of W generated by the simple reflections $r_j (j \in J)$ and let $W(J) = \{w \in W | w\Delta^- \cap \Delta^+ \subset \Delta^+(J)\}$. Then W_J is the Weyl group of the Kac-Moody algebra $g_0^{(J)}$ and $W(J)$ be the set of right coset representatives of W_J in W.

Proposition:[2]: $H_K^{(J)} = \bigoplus_{\substack{w \in W(J) \\ F \subset T \\ l(w)+|F|=k}} V_J(w(\rho - s(F)) - \rho)$ w

here $V_J(\mu)$ denotes the irreducible highest weight module over $g_0^{(J)}$ with heighest weight μ and F runs over all the finite subsets of T such that any two elements in F are mutually perpendicular. Here we denote by $|F|$, the number of elements in F and $s(F)$, the sum of elements in F.

Define the homology space $H^{(J)}$ of $g_-^{(J)}$ to be

$$H^{(J)} = \sum_{k=1}^{\infty} (-1)^{k+1} H_k^{(J)} = \sum_{\substack{w \in W(J) \\ F \subset T \\ l(w)+|F| \geq 1}} (-1)^{l(w)+|F|+1} V_J(w(\rho - s(F)) - \rho).$$

Let

$P(H^{(J)}) = \{\alpha \in Q^-(J) | dim H_\alpha^{(J)} \neq 0\} = \{\tau_1, \tau_2, \tau_3, \tau_4, \cdots\}$ and $d(i) = dinH_{\tau_i}^{(J)}$ for i=1,2,3,….

For $\tau \in Q^-(J)$, we denote by $T^{(J)}(\tau)$ the set of all partitions of τ into a sum of τ_i's,

(i.e)., $T^{(J)}(\tau) = \{n = (n_i)_{i \geq 1} | n_i \in \mathbb{Z}_{\geq 0}, \sum n_i \tau_i = \tau\}$.

For $n \in T^{(J)}(\tau)$, we will use the notaion $|n| = \sum n_i$ and $n! = \prod n_i!$. Now, for $\tau \in Q^-(J)$, we define a function

$$W^{(J)}(\tau) = \sum_{n \in T^{(J)}(\tau)} \frac{(|n|-1)!}{n!} \prod d(i)^{n_i}. \quad (1)$$

The function $W^{(J)}(\tau)$ is called the Witt partition function.

Theorem 1.1[2]: *Let $\alpha \in \Delta^-(J)$ be a root of a symmetrizable GKM algebra g. Then we have*

$$dim g_\alpha = \sum_{d|\alpha} \frac{1}{d} \mu(d) W(J)\left(\frac{\alpha}{d}\right)$$

$$= \sum_{d|\alpha} \frac{1}{d} \mu(d) \sum_{n \in T^{(J)}\left(\frac{\alpha}{d}\right)} \frac{(|n|-1)!}{n!} \prod d(i)^{n_i}$$

where μ is the classical Mobius function.

III. ROOT MULTIPLICITY OF A GKM ALGEBRAS OF FINITE A_4 TYPE

In this section, we explicitly determine the root multiplicities of GKM algebra associated with the GGCM which is an extension of finite A_4. Consider the GKM algebra of finite A_4 type associated with the Generalized Generalized Cartan Matrix (GGCM)

$$\begin{pmatrix} -k & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

of charge $\underline{m} = (s, 1,1,1,1)$ where $k, s \in \mathbb{Z}_{>0}$.

Let $I = \{1,2,3,4,5\}$ be the index set for the simple roots of g. Then, α_1 is the imaginary simple root with multiplicity $r \geq 1$ and $\alpha_2, \alpha_3, \alpha_4$ and α_5 are the real simple roots.

Then we have

$T = \{\alpha_1, \alpha_1, \cdots, \alpha_1\}$ counted s times.

Since $(\alpha_1, \alpha_1) = -k < 0$, F can be either empty or $\{\alpha_1\}$.
If we take $J = \{2,3,4,5\}$, then

$$g_0^{(J)} = g_0 \oplus \mathbb{C}h_1,$$

where $g_0 = \langle e_2, f_2, h_2, e_3, f_3, h_3, e_4, f_4, h_4, e_5, f_5, h_5 \rangle$
and $W(J) = \{1\}$
By proposition, we have

$$H_1^{(J)} = V_J(-\alpha_1) \oplus \cdots \oplus V_J(-\alpha_1) \ (s \text{ copies})$$
$$H_2^{(J)} = 0$$
$$\vdots$$
$$H_k^{(J)} = 0 \text{ for } k \geq 2.$$

Therefore we get

$$H^{(J)} = V_J(-\alpha_1) \oplus \cdots \oplus V_J(-\alpha_1) \ (s \text{ copies}),$$

where $V_J(-\alpha_1)$ is the standard representation.
By identifying $-l_1\alpha_1 - l_2\alpha_2 - l_3\alpha_3 - l_4\alpha_4 - l_5\alpha_5 \in Q^-$ with
$(l_1, l_2, l_3, l_4, l_5) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$, we have
$P(H^{(J)}) = \{(1,0,0,0,0), (1,1,0,0,0), (1,1,1,0,0), (1,1,1,1,0), (1,1,1,1,1)\}$, where

$$\dim H^{(J)}_{(1,0,0,0,0)} = \dim H^{(J)}_{(1,1,0,0,0)} = \dim H^{(J)}_{(1,1,1,0,0)} = \dim H^{(J)}_{(1,1,1,1,0)} = \dim H^{(J)}_{(1,1,1,1,1)} = s.$$

For $l_1, l_2, l_3, l_4, l_5 \in \mathbb{Z}_{\geq 0}$, the only partition of $(l_1, l_2, l_3, l_4, l_5)$ into a sum of $(1,0,0,0,0), (1,1,0,0,0), (1,1,1,0,0), (1,1,1,1,0), (1,1,1,1,1)$ is

$$T^{(J)}(\tau) = \{(n_1, n_2, n_3, n_4, n_5) = n_1(1,0,0,0,0) + n_2(1,1,0,0,0) + n_3(1,1,1,0,0) + n_4(1,1,1,1,0) + n_5(1,1,1,1,1) = (l_1, l_2, l_3, l_4, l_5)\}$$

Therefore, by solving the above condition we get,

$$(l_1, l_2, l_3, l_4, l_5) = (l_1 - l_2)(1,0,0,0,0) + (k_2 - k_3)(1,1,0,0,0) + (k_3 - k_4)(1,1,1,0,0) + (k_4 - k_5)(1,1,1,1,0) + k_5(1,1,1,1,1).$$

Thus the Witt partition function (1) becomes

$$W^{(J)}(l_1, l_2, l_3, l_4, l_5) = \frac{(l_1 - 1)!}{(l_1 - l_2)!(l_2 - l_3)!(l_3 - l_4)!(l_4 - l_5)!l_5!} r^{l_1}$$

Therefore, by the Theorem (1), we obtain the following proposition:

Proposition 3.1: Let $g = g(A, \underline{m})$ be the GKM algebra associated with the GGCM

$$A = \begin{pmatrix} -k & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

of charge $\underline{m} = (s, 1,1,1,1)$ with $k, s \in \mathbb{Z}_{>0}$.

Thus, for the root $\alpha = -l_1\alpha_1 - l_2\alpha_2 - l_3\alpha_3 - l_4\alpha_4 - l_5\alpha_5$ with $l_1, l_2, l_3, l_4, l_5 \in \mathbb{Z}_{\geq 0}$, we have

$$\dim g(l_1, l_2, l_3, l_4, l_5) = \frac{1}{k_1} \sum_{d \backslash (l_1, l_2, l_3, l_4, l_5)} \mu)d) \frac{\left(\frac{l_1 - 1}{d}\right)!}{\left(\frac{l_1 - l_2}{d}\right)!\left(\frac{l_2 - l_3}{d}\right)!\left(\frac{l_3 - l_4}{d}\right)!\left(\frac{l_4 - l_5}{d}\right)!\left(\frac{l_5}{d}\right)!} r^{\frac{l_1}{d}} \quad (2)$$

Some special cases:

1. If $l_5 > l_1$ or $l_4 > l_3$ or $l_3 > l_2$ or $l_2 > l_1$,

then $dim g_{(l_1,l_2,l_3,l_4,l_5)} = 0.$

2. If $l_1 = l_2 = l_3 = l_4 = l_5$,

then $dim g_{(l_1,l_1,l_1,l_1,l_1)} = \frac{1}{k_1} \sum_{d|l_1} \mu(d) r^{\frac{k_1}{d}}$.

Example:

(i) For the root (6,5,4,3,2), using the formula (2), we have

$$\dim g_{(6,5,4,3,2)} = \frac{1}{6} \times \frac{6!}{4!2!} \times \frac{4!}{3!1!} \times \frac{3!}{2!1!} \times \frac{2!}{1!1!} r^6 = 60r^6$$

In particular when r=2, we get $dim g_{(6,5,4,3,2)} = 1920$

(ii) For the root (10,8,6,4,2), using the formula (2), we have

$$\dim g_{(10,8,6,4,2)} = 11340r^{10} - 12r^5$$

In particular when r=2, we get $\dim g_{(10,8,6,4,2)} = 11611776$

(iii) For the root (3,3,3,3,3), using the formula (2), we have

$$\dim g_{(3,3,3,3,3)} = \frac{r^3 - r}{3}$$

In particular when r=3, we get $\dim g_{(3,3,3,3,3)} = 8$

(iv) For the root (5,4,3,2,1), using the formula (2), we have

$$\dim g_{(3,3,3,3,3)} = 12r^5$$

In particular when r=5, we get $\dim g_{(3,3,3,3,3)} = 37500$

IV. CONCLUSION

Thus, we have delimited the root multiplicity for the finite type of GKM algebras A_4. Similarly we can explicitly find the root multiplicity for other families of finite, affine and hyperbolic type of GKM algebras.

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