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# Continuous-Time Linear Parameter Varying System Identification With Fractional Models

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**--------------------------------------------------------*ABSTRACT*----------------------------------------------------------**
*This paper deals with continuous-time identification of linear parameter varying systems with fractional models using a direct approach. Methods based on the least squares and instrumental variables approaches are firstly proposed. Then an optimization approach combined with the optimal instrumental variables algorithm is presented to identify the commensurate order of the fractional model. A numerical example is investigated in order to assess the consistency of the developed methods in a noisy output context and via a Monte Carlo simulation study.*
***KEYWORDS*** *- Continuous-time, commensurate order, fractional differentiation, system identification, LPV system.*

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## I INTRODUCTION

The non-integer order derivative has attracted a considerable attention recently. Its major characteristics are the ability to describe the complex behavior of many physical systems and to model high order integer complex systems with a reduced number of parameters. Therefore, fractional differentiation has been used in various fields. In biology, where macro-scale behavior are predicted from micro-scale observations using fractional calculus [1]. In chemical physics, it was proven that the relation between the heat flow and the temperature in a semi-infinite homogeneous medium depends on half-order fractional derivatives [2]. Also, it is a well-known tool for controller synthesis such as the fractional PID controller [3] and the CRONE controller [4].

Concerning the identification framework, fractional differentiation has been the subject of many types of research. Various methods based on fractional linear models have been developed. Only the Linear Time Invariant (LTI) context has been considered. Refer to [5, 6, 7, 8, 9, 10] for an overview. Nonetheless, linear models are not accurate enough for physical system modeling since, in practice, physical behaviors may present nonlinearity and/or a time varying nature. In [11], fractional nonlinear models based on Volterra series are developed and proposed for system identification. To the best of author's knowledge, there exist no identification methods for fractional systems with time varying nature except the work in [12]. In the latter study, the authors have proved that a thermal system that exhibits a diffusive interface have a fractional comportment. Also, it presents nonlinear properties due to the dependency of the system dynamics on the temperature. So to identify this system, a fractional continuous-time linear parameter varying (LPV) model is used.

Linear Parameter Varying systems are a class of dynamical systems whose mathematical description depends on an external parameter, named the scheduling variable, which change values over time. This class of system envelops a wide variety of systems encountered in practice [13].

Recently, many types of research have been interested in continuous-time (CT) and discrete-time (DT) LPV system identification with rational models. In the available studies, two approaches are established to identify a CT LPV system: the first one is the local approach where, at each working point, a local LTI model is identified. Then, the LPV model is obtained by interpolating the obtained linear models [14, 15]. The second one is the global approach where a variable scheduling parameter is considered to identify the LPV model [16, 17,18, 19]. In this work, the global approach is considered.

Our main contribution is to extend CT system identification methods with LPV rational models to the fractional case basing on a global approach. The developed methods are based on the Least Squares (LS) techniques and called fractional-linear parameters varying-ordinary least squares fLPV- OLS and the instrumental variable (IV) techniques and called fractional linear parameters varying-instrumental variable fLPV-IV and fractional-linear parameters varying-simplified refined instrumental variable fLPV-SRIV. Then, a gradient-based approach combined to the fLPV-SRIV estimator allows the estimation of the fractional differentiation orders.

This paper is outlined in the following way. The next Section details the differential equation representation of CT fractional LPV systems. In Section 3, 4 and 5 the proposed methods for CT fractional LPV systems are detailed. Their performances are analyzed in Section 6 through a Monte Carlo simulation study. Finally, Section 7 concludes the paper and indicates some future work.

## II FRACTIONAL LPV SYSTEMS

### 1.1. Mathematical Background

The SISO LPV system input $u(t)$ and the noise-free output $y_0(t)$ are related by the following fractional differential equation:

$$y_0(t)+\sum_{n=1}^{N} a_n(\rho_t)\, p^{\alpha_n} y_0(t) = \sum_{m=0}^{M} b_m(\rho_t)\, p^{\beta_m} u(t) \qquad (1)$$

where $p$ is the time-domain differential operator (also denoted $D$), $p = D = \frac{d}{dt}$ and the differentiation orders

$$\alpha_1 < \alpha_2 < \cdots < \alpha_N ; \beta_0 < \beta_1 < \cdots < \beta_M$$

are allowed to be non-integer positive numbers and ordered for identifiability purposes.

$\rho_t : \mathbb{R} \to \mathrm{P}$ (the compact $\mathrm{P} \in \mathbb{R}^{n_{\mathrm{P}}}$ denotes the scheduling space) is the scheduling variable, $\rho_t = \rho(t)$.

The coefficients of the fractional differential equation $a_n(\rho_t)$ and $b_m(\rho_t)$ are functions with static dependence on the scheduling variable $\rho_t$ at time $t$, i.e dependence only on the instantaneous value of $\rho_t$ at time $t$.

To compute fractional derivatives of a continuous time function, different definitions are developed [20, 21, 22, 5]. The $\upsilon$-order fractional derivative of a continuous-time function $f(t)$, relaxed at $t = 0$, i.e. $f(t) = 0$ for $t \leq 0$, is numerically evaluated using the Grünwald approximation [23]:

$$p^{\upsilon} f(t) \approx \frac{1}{h^{\upsilon}} \sum_{k=0}^{K} (-1)^k \binom{\upsilon}{k} f(t-kh), \;\; \forall t \in \mathbb{R}_+^* \qquad (2)$$

$\upsilon \in \mathbb{R}_+^*$, $K = \left\lfloor \frac{t}{h} \right\rfloor$ with $\lfloor . \rfloor$ denotes the floor operator, $h$ is the sampling period and $\binom{\upsilon}{k}$ is the Newton's binomial generalized to fractional orders:

$$\binom{\upsilon}{k} = \begin{cases} 1 & \text{if } k = 0 \\ \dfrac{\upsilon(\upsilon-1)(\upsilon-2)\cdots(\upsilon-k+1)}{k!} & \text{if } k > 0 \end{cases} \qquad (3)$$

For fractional LPV systems time-domain simulation, the Grünwald approximation (equation (2)) is used.

### 1.2. Problem Statement

Consider the noisy fractional LPV SISO system described by Figure 1.

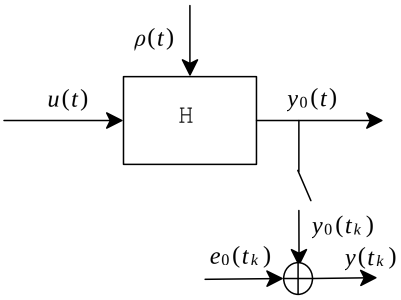



Figure 1: Fractional LPV system in a noisy output context.

The continuous-time fractional system $H$, in a noisy output context, is described by the following LPV input-output representation with a static scheduling dependence:

$$\begin{cases} A(\rho_t, p)y_0(t) = B(\rho_t, p)u(t) \\ y(t_k) = y_0(t_k) + e_0(t_k) \end{cases} \quad (4)$$

where $t_k = kh$ ($k \in \square$ ). $y(t_k)$ is the measured output signal and $e_0(t_k)$ is its additive noise.

The $\rho_t$ dependent polynomials $A$ and $B$ are defined by:

$$\begin{cases} A(\rho_t, p) = 1 + \sum_{n=1}^{N} a_n(\rho_t) p^{\alpha_n} \\ B(\rho_t, p) = \sum_{m=0}^{M} b_m(\rho_t) p^{\beta_m} \end{cases} \quad (5)$$

where $a_n(\rho_t)$ and) are parameterized as:

$$\begin{cases} a_n(\rho_t) = a_{n,0} + \sum_{l=1}^{L_f} a_{n,l} f_l(\rho_t); n = 1, ..., N \\ b_m(\rho_t) = \sum_{l=0}^{L_g} b_{m,l} g_l(\rho_t); m = 0, ..., M \end{cases} \quad (6)$$

$\{f_l\}_{l=1}^{L_f}$ and $\{g_l\}_{l=1}^{L_g}$ are assumed to be *a priori* known meromorphic function ( $f$ is called a meromorphic function if it takes the form $f = \frac{\kappa}{\kappa_1}$ with $\kappa$ and $\kappa 1$ are analytic functions and $\kappa 1 \neq 0$ .). This class of functions contains the common functional dependencies that result during LPV modeling of physical systems [16].

For simplification, $a_n(\rho_t)$ and $B_m(\rho_t)$ are supposed to be function of the same $\rho_t$ -function $f_l$ defined as follows:

$$f_l(\rho_t) = \rho_t^{\,l},\ l = 0, \cdots, L \quad (7)$$

A polynomial dependence on $\rho_t$ for the coefficients is considered:

$$\begin{cases} a_n(\rho_t) = a_{n,0} + a_{n,1}\rho_t + \cdots + a_{n,L}\rho_t^L; n = 1, \cdots, N \\ b_m(\rho_t) = b_{m,0} + b_{m,1}\rho_t + \cdots + b_{m,L}\rho_t^L; m = 0, \cdots, M \end{cases} \quad (8)$$

Let $\theta$, the parameters vector containing the coefficients of the fractional differential equation:

$$\theta = [a_1, \cdots, a_N, b_0, \cdots, b_M]^T \quad (9)$$

with

$a_n = [a_{n,0}, a_{n,1}, \cdots, a_{n,L}]$; $n = 1, \cdots, N$

(10)

$b_m = [b_{m,0}, b_{m,1}, \cdots, b_{m,L}]$; $m = 0, \cdots, M$

$D_{N_t} = \{y(t_k), u(t_k), \rho(t_k)\}_{k=0}^{N_t}$ ( $N_t$ is the number of samples) denotes a set of available data sampled with a sampling period $h$.

Our objective is to estimate $\theta$ using $D_{Nt}$. Firstly the fractional orders are fixed *a priori* and only the fractional differential equation coefficients are estimated. Then, the case of unknown fractional orders and the differential equation coefficients is treated.

## III IDENTIFICATION METHODS FOR FRACTIONAL LPV SYSTEMS

### 3.1. *SVF BASED* METHODS

The estimated output signal $\hat{y}(t_k)$ is written in a linear regression form:

$$\hat{y}(t_k) = \Phi^T(t_k)\theta \quad (11)$$

where the regression vector $\Phi^T(t_k)$ is defined by:

$$\Phi(t_k) = \varphi(t_k)^T \otimes F \quad (12)$$

$\otimes$ denotes the kronecker product (or tensor product), $F$ is defined by:

$F = [1, f_1(\rho),\cdots,f_L(\rho)]$ (13)

and $\varphi^T(t_k)$ is given by:

$$\varphi^T(t_k) = \left[-p^{\alpha_1}y(t_k),\cdots,-p^{\alpha_N}y(t_k), p^{\beta_0}u(t_k),\cdots, p^{\beta_M}u(t_k)\right] \quad (14)$$

The regression vector defined by (14) contains fractional derivatives of the input and the noisy output signals.

Note that fractional differentiation is characterized by a long memory. Also, the use of the Grünwald approximation (equation (2)), to compute the fractional differentiation, amplifies the additive noise effect. To solve this problem, the use of a low pass filter is proposed. The filter differentiates signals at low frequencies and filter high frequencies. An extension of the State Variable Filter (SVF) approach for fractional derivatives is proposed in [21, 24] and used in this work.

The transfer function of the fractional SVF $F_\upsilon(s)$ is defined by the following expression:

$$F_\upsilon(s) = s^\upsilon\left(\frac{\lambda}{\lambda+s}\right)^{\lfloor\alpha_N\rfloor+1} \quad (15)$$

where $s$ is the Laplace variable, $\lambda$ is the cut-off frequency and $\lfloor . \rfloor$ stands for the floor operator.

Thus, the regression vector (14) can be rewritten as:

$$\varphi_f^{\,T}(t_k) = \left[-p^{\alpha_1}y_f(t_k),\cdots,-p^{\alpha_N}y_f(t_k), p^{\beta_0}u_f(t_k),\cdots, p^{\beta_M}u_f(t_k)\right] \quad (16)$$

where $p^{\alpha n}y_f(t_k)$ and $p^{\beta m}u_f(t_k)$ are the fractional filtered derivatives of the input and the measured output signals:

$$\begin{cases} p^{\alpha_n}y_f(t_k) = F_{\alpha_n}(s)y(t_k); 1\le n\le N \\ p^{\beta_m}u_f(t_k) = F_{\beta_m}(s)u(t_k); 0\le m\le M \end{cases} \quad (17)$$

The main idea is to use the filtered signals instead of noisy ones to estimate the parameters vector.

#### 3.1.1. FRACTIONAL-LINEAR PARAMETER VARYING-ORDINARY LEAST SQUARES ALGORITHM (FLPV-OLS)

This method is inspired by the work developed for DT LPV system identification with rational models [17] and extended in this section for fractional models.

The *fLPV-OLS* estimator is given by:

$$\hat{\theta}_{fLPV-OLS} = \arg\min_{\theta\in\mathbb{R}^{n_\theta}} V(D_{N_t},\theta) \quad (18)$$

where $n_\theta$ is the number of parameters to be estimated and the cost function $V(D_{Nt},\theta)$ is defined as:

$$V(D_{N_t},\theta) = \frac{1}{N_t}\sum_{k=1}^{N_t}\frac{1}{2}e_\theta^{\,2}(t_k) \quad (19)$$

and based on the equation error:

$$e_\theta(t_k) = y_f(t_k) - \Phi_f^T(t_k)\theta \quad (20)$$

Then, the optimal estimator is given by:

$$\hat{\theta}_{fLPV-OLS} = \left[\frac{1}{N_t}\sum_{k=1}^{N_t}\Phi_f(t_k)\Phi_f^{\,T}(t_k)\right]^{-1}\left[\frac{1}{N_t}\sum_{k=1}^{N_t}\Phi_f(t_k)y_f(t_k)\right] \quad (21)$$

This estimator is unbiased and consistent if:

$$\lim_{N_t\to\infty} e_{\theta=0} \quad (22)$$

In the noisy framework, the *fLPV-OLS* estimator gives biased parameters. Particularly in the case of fractional LPV system identification, fractional derivatives take into account the whole past of the noisy output and the linear parameters variation. To obtain an unbiased estimation of the parameters vector, an instrumental variable approach is proposed and presented in the next section.

### 3.1.2. FRACTIONAL-LINEAR PARAMETER VARYING-INSTRUMENTAL VARIABLE ALGORITHM (FLPV-IV)

The DT LPV system identification with rational models, developed in [25] is extended for CT LPV system identification with fractional models in this section.

It is based on an auxiliary model obtained by computing an estimation using the ordinary least squares algorithm. It consists in introducing a new vector $\zeta(t_k)$ called the instrument. $\zeta(t_k)$ must be correlated with the regression vector $\Phi(t_k)$, defined by (12) and decorrelated with the additive noise.

Then, the IV estimator based on the fractional SVF approach is given by:

$$\hat{\theta}_{fLPV-IV} = \left[\frac{1}{N_t}\sum_{k=1}^{N_t}\zeta_f(t_k)\Phi_f^T(t_k)\right]^{-1}\left[\frac{1}{N_t}\sum_{k=1}^{N_t}\zeta_f(t_k)y_f(t_k)\right] \quad (23)$$

The IV vector $\zeta_f$ is defined by:

$$\zeta_f^{\,T}(t_k) = \tilde{\varphi}_f \otimes F \quad (24)$$

with

$$\tilde{\varphi}_f^{\,T}(t_k) = \left[-p^{\alpha_1}\tilde{y}_f(t_k),\cdots,-p^{\alpha_N}\tilde{y}_f(t_k), p^{\beta_0}u_f(t_k),\cdots,p^{\beta_M}u_f(t_k)\right] \quad (25)$$

where $p^{\alpha_n}\tilde{y}_f$ and $p^{\beta_n}u_f$ are respectively the fractional filtered derivatives of the auxiliary model output and the input signals.

Two conditions must be fulfilled to guarantee the convergence of this algorithm:

$$\begin{cases} E\left[\zeta_f(t_k)\Phi_f^{\,T}(t_k)\right] \text{is non singular,} \\ E\left[\zeta_f(t_k)e_0(t_k)\right] = 0 \end{cases} \quad (26)$$

where $E$ [.] denotes the mathematical expectation.

Note that, the IV algorithm typically gives unbiased estimates with a large estimate variance [25]. Next, to ameliorate these results, iterative techniques are proposed.

## 3.2. FRACTIONAL-LINEAR PARAMETER VARYING- SIMPLIFIED REFINED INSTRUMENTAL VARIABLE ALGORITHM (FLPV-SRIV)

The simplified refined instrumental variable approach (SRIV) is developed to identify a CT rational LPV systems [19] and for CT LTI system identification with fractional models [26].

A generalization of the SRIV approach for CT fractional LPV systems identification is proposed [27].

Substituting the linear parameters by their expressions given by (5) and (6) in the fractional LPV system (equation (4)) yields to:

$$\begin{cases} y_0(t) + \sum_{n=1}^{N} a_{n,0}p^{\alpha_n}y_0(t) = -\sum_{n=1}^{N}\sum_{l=1}^{L} a_{n,l}f_l(\rho_t)p^{\alpha_n}y_0(t) + \\ \qquad \sum_{m=0}^{M}\sum_{l=0}^{L} b_{m,l}f_l(\rho_t)p^{\beta_m}u(t) \\ y(t_k) = y_0(t_k) + e_0(t_k) \end{cases} \quad (27)$$

Equation (27) can be rewritten as:

$$\begin{cases} F_0(p)y_0(t) = -\sum_{n=1}^{N}\sum_{l=1}^{L} a_{n,l}y^{\alpha_n,l}(t) + \sum_{m=0}^{M}\sum_{l=0}^{L} b_{m,l}u^{\beta_m,l}(t) \\ y(t_k) = y_0(t_k) + e_0(t_k) \end{cases} \quad (28)$$

(28)

where

$$\begin{cases} F_0(p)y_0(t) = -\sum_{n=1}^{N}\sum_{l=1}^{L} a_{n,l}y^{\alpha_n,l}(t) + \sum_{m=0}^{M}\sum_{l=0}^{L} b_{m,l}u^{\beta_m,l}(t) \\ y(t_k) = y_0(t_k) + e_0(t_k) \end{cases} \quad (29)$$

and

$$\begin{cases} y^{\alpha_n,l}(t) = f_l(\rho_t)p^{\alpha_n}y_0(t); \{n,l\} \in \{1,...,N;1,...,L\} \\ u^{\beta_m,l}(t) = f_l(\rho_t)p^{\beta_m}u(t); \{m,l\} \in \{0,...,M;0,...,L\} \end{cases} \quad (30)$$

Equation (27) can be rewritten as follows:

$$\begin{cases} y_0(t) = -\sum_{n=1}^{N}\sum_{l=1}^{L} \frac{a_{n,l}}{F_0(p)} y^{\alpha_n,l}(t) + \sum_{m=0}^{M}\sum_{l=0}^{L} \frac{b_{m,l}}{F_0(p)} u^{\beta_m,l}(t) \\ y(t_k) = y_0(t_k) + e_0(t_k) \end{cases} \quad (31)$$

Then, the new estimated output can be written as:

$$\hat{y}(t_k) = \varphi_f^T(t_k)\theta \quad (32)$$

The regression vector is defined as:

$$\begin{aligned} \varphi_f^{\,T}(t_k) = \big[&-y_f^{\alpha_1,1}(t_k),\cdots,-y_f^{\alpha_N,1}(t_k),\cdots,-y_f^{\alpha_1,L}(t_k),\cdots,-y_f^{\alpha_N,L}(t_k), \\ &u_f^{\beta_0,1}(t_k),\cdots,u_f^{\beta_M,1}(t_k),\cdots,u_f^{\beta_0,L}(t_k),\cdots,u_f^{\beta_M,L}(t_k)\big] \end{aligned} \quad (33)$$

$u_f^{\beta_m,l}$ and $y_f^{\alpha_n,l}$ are the fractional derivatives of the filtered input and output signals.

Let $Q_0^i(s)$ denotes the filter transfer function which depends on the estimates at the iteration *i*.

$$Q_0^i(s) = \frac{1}{F_0^i(s)} = \frac{1}{1+\sum_{n=1}^{N}\hat{a}_{n,0}^i s^{\alpha_n}} \quad (34)$$

By assuming this problem reformulation, the *fLPV-SRIV* estimator is defined by:

$$\hat{\theta}_{fLPV-SRIV}^i = \arg\min_{\theta\in\mathbb{R}^{n_\theta}} V^i(D_{N_t},\theta) \quad (35)$$

where $V^i(D_{N_t},\theta)$ is the cost function minimized at each iteration:

$$V^i(D_{N_t},\theta) = \frac{1}{N_t}\sum_{n=1}^{N_t}\left(e_\theta^i(t_k)\right)^2 \quad (36)$$

$e_\theta^i(t_k)$ is the equation error.

Then, the optimal solution is given by:

$$\hat{\theta}_{fLPV-SRIV}^i = \left[\frac{1}{N_t}\sum_{k=1}^{N_t}\zeta_f(t_k)\varphi_f^T(t_k)\right]^{-1}\left[\frac{1}{N_t}\sum_{k=1}^{N_t}\zeta_f(t_k)y_f(t_k)\right] \quad (37)$$

The *fLPV-SRIV* algorithm is summarized by:

**Step 1**: ***i* = 0**

Compute the first estimate by applying the f*LPV-IV* estimator (equation (23)).

$$\hat{\theta}_{fLPV-SRIV}^{\,0} = \hat{\theta}_{fLPV-IV}$$

**Step 2**: ***i=i+1***

compute $\tilde{y}(t_k)$ the output signal using the obtained model at the previous iteration.

**Step 3:**

compute the estimated continuous-time filter :

$$Q_0^i(s) = \frac{1}{F_0^i(s)} = \frac{1}{1+\sum_{n=1}^{N}\hat{a}_{n,0}^i s^{\alpha_n}} \quad (38)$$

and generate the filtered fractional derivatives of the input and the output signals $u_f^{\beta m,l}$ and $\tilde{y}_f^{\alpha_n,l}$ .

**Step 4:** built the filtered regression vector $\phi_f(t_k)$ and the filtered instrumental vector $\zeta_f(t_k)$:

$$\varphi_f^{\,T}(t_k)=\Big[-y_f^{\alpha_1,1}(t_k),\cdots,-y_f^{\alpha_N,1}(t_k),\cdots,-y_f^{\alpha_1,L}(t_k),\cdots,-y_f^{\alpha_N,L}(t_k),\; u_f^{\beta_0,1}(t_k),\cdots,u_f^{\beta_M,1}(t_k),\cdots,u_f^{\beta_0,L}(t_k),\cdots,u_f^{\beta_M,L}(t_k)\Big] \quad (39)$$

$$\zeta_f^{\,T}(t_k)=\Big[-\tilde{y}_f^{\alpha_1,1}(t_k),\cdots,-\tilde{y}_f^{\alpha_N,1}(t_k),\cdots,-\tilde{y}_f^{\alpha_1,L}(t_k),\cdots,-\tilde{y}_f^{\alpha_N,L}(t_k),\; u_f^{\beta_0,1}(t_k),\cdots,u_f^{\beta_M,1}(t_k),\cdots,u_f^{\beta_0,L}(t_k),\cdots,u_f^{\beta_M,L}(t_k)\Big] \quad (40)$$

and compute the *fLPV-SRIV* estimate $\hat{\theta}^i_{fLPV-SRIV}$ using equation (37);

**Step 5:**
if convergence occurs according to a specified convergence criterion

$$i \quad \frac{\left(\hat{\theta}^i_{fLPV-SRIV}-\hat{\theta}^{i-1}_{fLPV-SRIV}\right)^T\left(\hat{\theta}^i_{fLPV-SRIV}-\hat{\theta}^{i-1}_{fLPV-SRIV}\right)}{\left(\hat{\theta}^{i-1}_{fLPV-SRIV}\right)^T\left(\hat{\theta}^{i-1}_{fLPV-SRIV}\right)}<\epsilon; \quad \text{;where } \epsilon=10^{-5} \;--$$

or a maximum number of iterations is reached, then stop, else go to step2.

## IV COMMENSURATE FRACTIONAL ORDER AND LINEAR COEFFICIENTS ESTIMATION

In this section, the fractional orders of the LPV model are assumed unknown. An algorithm called order optimization-*fLPV-SRIV* (*OO-fLPVSRIV*) is proposed to compute both coefficients and fractional orders estimates. The new estimated parameters vector is given by:

$\Theta = [\theta,\eta]^T$ (41)

where

$\theta = [a_1,\cdots,a_N,b_0,\cdots,b_M]$

(42)

$\eta = [\alpha_1,\cdots,\alpha_N,\beta_0,\cdots,\beta_M]$

The number of the parameters to be estimated is $(L + 2)(N + M + 1)$. To be reduced, considering a commensurate order model described by:

$$y(t)+\sum_{n=1}^{N'}a_n(\rho_t)D^{n\upsilon}y(t)=\sum_{m=0}^{M'}b_m(\rho_t)D^{m\upsilon}u(t) \quad (43)$$

where $\upsilon$ is the commensurate order (the commensurate order $\upsilon$ is the biggest real number such that all differentiation orders are integer multiples of $\upsilon$. $\upsilon$ exist assuming that all differential orders are rational (in $\mathbb{Q}^*$) and where $N'=\dfrac{\alpha_N}{\upsilon}$ and $M'=\dfrac{\beta_M}{\upsilon}$ are integers.

Thus, the new parameters vector is given by:

$\Theta = [a_1,\cdots,a_N,b_0,\cdots,b_M,\upsilon]^T$ (44)

So, the commensurate order is estimated by minimizing the following quadratic criterion:

$$J(\hat{\Theta})=\frac{1}{2}\left\|\varepsilon_\Theta\right\|^2 \quad (45)$$

where $\varepsilon_\Theta$ is the output error given by:

$$\varepsilon_\Theta(t_k)=y(t_k)-\hat{y}(t_k) \quad (46)$$

The quadratic criterion (equation (45)) is nonlinear to Θ. Then a nonlinear optimization technique is required to iteratively estimate Θ. The Gauss-Newton algorithm is chosen in this work.

The optimization order-*fLPV-SRIV* algorithm (*OO-fLPV-SRIV*) is summarized as follows:

**Step 1: *i* = 0 Initialization**

initialize $\upsilon = \upsilon^0$ and estimate the fractional differential equation coefficients using the *fLPV-SRIV* algorithm.

**Step 2**: **GAUSS-NEWTON OPTIMIZATION METHOD**

**do**

Set $\lambda = \Lambda$ ($\lambda$ is a positive real constant)
do

- refine the new commensurate order:

$$\varepsilon_\Theta(t_k) = y(t_k) - \hat{y}(t_k)\upsilon^{i+1} = \upsilon^i - \lambda H^{-1} \left.\frac{\partial J^i}{\partial \upsilon}\right|_{\upsilon^i} \quad (47)$$

$$\begin{cases} \dfrac{\partial \varepsilon_\Theta}{\partial \upsilon} = \left[ -\dfrac{\partial \varepsilon_\Theta}{\partial \alpha_1}^T, \cdots, -\dfrac{\partial \varepsilon_\Theta}{\partial \alpha_N}^T, -\dfrac{\partial \varepsilon_\Theta}{\partial \beta_0}^T, \cdots, -\dfrac{\partial \varepsilon_\Theta}{\partial \beta_M}^T \right] : \text{the error sensitivity} \\ \dfrac{\partial J}{\partial \upsilon} = \dfrac{\partial \varepsilon_\Theta}{\partial \upsilon}^T \varepsilon_\Theta : \text{the gradient} \\ H = \dfrac{\partial \varepsilon_\Theta}{\partial \upsilon}^T \dfrac{\partial \varepsilon_\Theta}{\partial \upsilon} : \text{the approximate hesssian} \end{cases} \quad (48)$$

- compute the fractional differential equation coefficients via the *fLPVSRIV* algorithm.
- evaluate $J^{i+1}$
- $\lambda = \lambda/2$

while $J^{i+1} > J^i$
$i = i + 1$
**while**

$$\frac{\left(\hat{\theta}^i_{OO-fLPV-SRIV} - \hat{\theta}^{i-1}_{OO-fLPV-SRIV}\right)^T \left(\hat{\theta}^i_{OO-fLPV-SRIV} - \hat{\theta}^{i-1}_{OO-fLPV-SRIV}\right)}{\left(\hat{\theta}^{i-1}_{OO-fLPV-SRIV}\right)^T \left(\hat{\theta}^{i-1}_{OO-fLPV-SRIV}\right)} > \grave{o};$$

$\grave{o} = 10^{-5}$ or a maximum number of iteration is not reached.

## V NUMERICAL EXAMPLE

In the first part of this section, the fractional orders are fixed *a priori* and only the coefficients are estimated. In the second part, the commensurate order is assumed unknown and estimated along with the coefficients.

Considering the following fractional LPV model:

$$(S): \begin{cases} A(\rho_t, p) = 1 + a_1(\rho_t)p^\upsilon + a_2(\rho_t)p^{2\upsilon} \\ B(\rho_t, p) = b_0(\rho_t)\% + b_1(\rho_t)p^{\beta_1} \end{cases} \quad (49)$$

where

$$\begin{cases} a_1(\rho_t) = a_1^0 + a_1^1\rho = 1 - \rho_t \\ a_2(\rho_t) = a_2^0 + a_2^1\rho = 2 + \rho_t \\ b_0(\rho_t) = b_0^0 + b_0^1\rho = 2 + \rho_t \\ \%b_1(\rho_t) = b_1^0 + b_1^1\rho = 2 + 2\rho_t \end{cases} \quad (50)$$

and the scheduling signal $\rho_t$ is defined by:

$$\rho_t = \rho(t) = \sin(\frac{2\pi}{100}t) \quad (51)$$

The sampling period $h = 0.1$ sec. The input signal $\mathrm{U}(-1,1)$ is a uniformly distributed sequence (Figure 2). The measured output signal is corrupted by an additive noise (Figure 2).

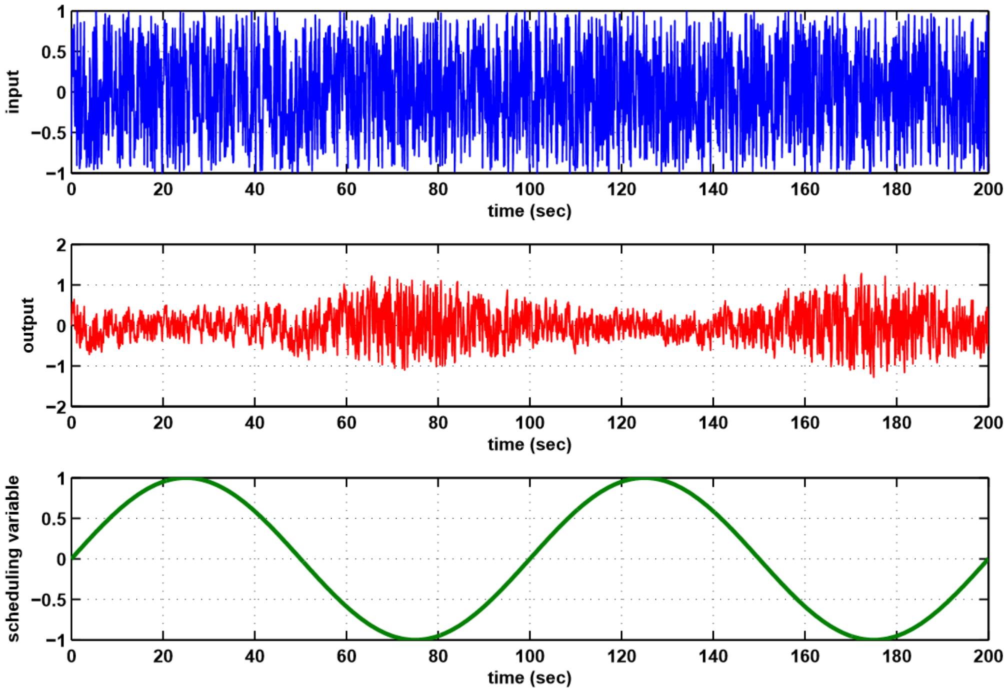



**Figure 2: Input, noisy output and scheduling variable signals zoomed between $t$ = 0 and 200 sec** ($SNR$ = 25 dB).

### 5.1. KNOWN ORDER

In this section, the fractional commensurate order is *a priori* fixed to $\upsilon = 0.75$. The parameters vector is defined by:

$\theta_0 = [1,-1,2,1,2,1]^T$ (52)

First, the choice of the SVF cut-off frequency and the number of data samples are studied with the help of Monte Carlo simulation. Secondly, a comparison study is made between the developed methods. Thirdly, the *SNR* (Signal to Noise Ratio) influence on the *fLPV-SRIV* quality of estimation is analyzed. Finally, the choice of the fractional order is discussed.

#### 5.1.1. CHOICE OF THE SVF CUT-OFF FREQUENCY

To study the influence of the cut-off frequency $\lambda$ on the quality of estimation, $\lambda$ is chosen between 0.1 rad/sec and 60 rad/sec. For each value of $\lambda$, the three developed algorithms are applied. $N_t = 4000$ is the number of samples and $N_{mc} = 300$ runs of Monte Carlo simulations ($SNR$ = 20 dB) are applied.

The normalized relative quadratic error ($NRQE$) defined by:

$$NRQE = \sqrt{\frac{1}{N_{mc}} \sum_{i=1}^{N_{mc}} \frac{\left\| \hat{\theta}_i - \theta_0 \right\|^2}{\left\| \theta_0 \right\|^2}} \quad (53)$$

is plotted in Figure 3. Note that the *fLPV-OLS* is the most influenced by the choice of $\lambda$. For the next, $\lambda_{opt} = 20$ rad/sec will be considered.

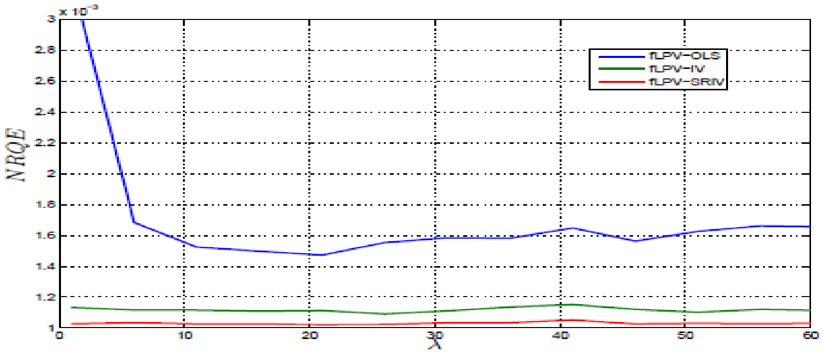



**Figure 3: Evolution of the *NRQE* according to the SVF cut-off frequency $\lambda$.**

### 5.1.2. Comparative study

In this part, a comparative study between the developed identification estimators is analyzed. $N_t$ = 4000 samples are collected. The filter cut-off frequency is set to $\lambda$ = 20 rad/sec. $N_{mc}$ = 300 runs of Monte Carlo simulation with different white noise realizations for *SNR* = 20 dB are considered.

For each Monte Carlo run, the *fLPV-OLS*, the *fLPV-IV* and *fLPV-SRIV* algorithms are applied. Table 1 illustrates the mean, the standard deviation (std) of each estimated parameter and the *NRQE*.

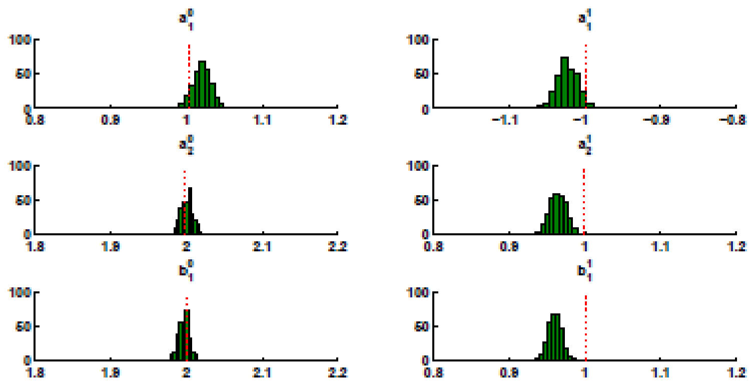
From the obtained results it can be noticed that the *fLPV-OLS* algorithm gives biased estimates unlike the *fLPV-IV* and the *fLPV-SRIV* algorithms.

**Table 1: Monte Carlo simulation results for *SNR* = 20 dB: Comparative study.**

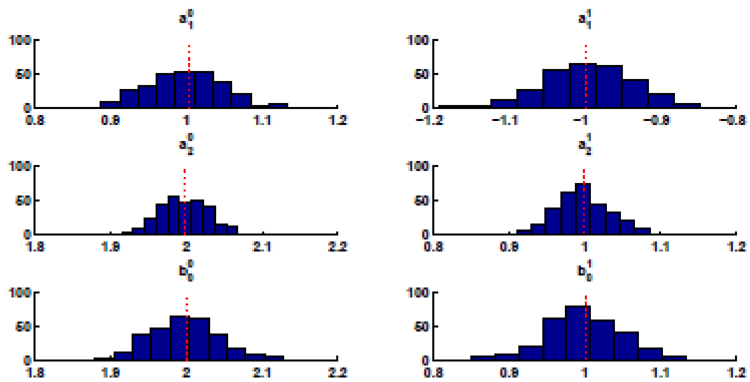
| Method | fLPV-OLS | | | fLPV-IV | | | fLPV-SRIV | | |
|---|---|---|---|---|---|---|---|---|---|
| Parameter | Mean | std | NRQE | Mean | std | NRQE | Mean | std | NRQE |
| $a_1^0 = 1$ | -0.2813 | 0.0065 | | 1.0206 | 0.0499 | | 0.9996 | 0.0115 | |
| $a_1^1 = -1$ | -0.2885 | 0.0083 | | -1.0223 | 0.0609 | | -0.9946 | 0.0136 | |
| $a_2^0 = 2$ | 0.5450 | 0.0063 | 0.8013 | 2.0009 | 0.0303 | 0.0329 | 1.9966 | 0.0073 | 0.01889 |
| $a_2^0 = 1$ | 0.3674 | 0.0144 | | 0.9648 | 0.0341 | | 0.9982 | 0.0109 | |
| $b_0^0 = 2$ | 0.4324 | 0.0085 | | 1.9969 | 0.0455 | | 1.9974 | 0.0069 | |
| $b_0^1 = 1$ | 0.2375 | 0.0145 | | 0.9612 | 0.0511 | | 0.9979 | 0.0087 | |

Also, the *fLPV-SRIV* estimator provides more consistent estimates with a minimal variance.

The histograms of the *fLPV-IV* and *fLPV-SRIV* estimates are depicted in Figure 4.



(a) *fLPV-IV*



(b) *fLPV-SRIV*

Figure 4: Distribution of estimates for *SNR* = 20 dB: Comparison study.

The *fLPV-SRIV* estimates are around the true values, which prove their efficiency. For the rest of the study, only the *fLPV-SRIV* will be considered.

5.1.3. SNR INFLUENCE

Different white noise realizations are considered ($SNR$ = 10 dB) with $N_{mc}$ = 300. Simulation results of the *fLPV-SRIV* are illustrated in the Table 2.

**Table 2: Simulation results for *SNR* = 10 Db**

| Parameter | Mean | std | NRQE |
|---|---|---|---|
| $a_1^0 = 1$ | 0.9945 | 0.0217 | |
| $a_1^1 = -1$ | -1.0261 | 0.0252 | |
| $a_2^0 = 2$ | 2.0039 | 0.0160 | 0.0259 |
| $a_2^0 = 1$ | 0.9916 | 0.0212 | |
| $b_0^0 = 2$ | 1.9970 | 0.0111 | |
| $b_0^1 = 1$ | 0.9903 | 0.0134 | |

Figure 5 illustrates the histograms of the *fLPV-SRIV* estimates. Even, in the case of a high level noisy context the *fLPV-SRIV* algorithm still gives unbiased estimates, which improve the effectiveness of the developed estimator.

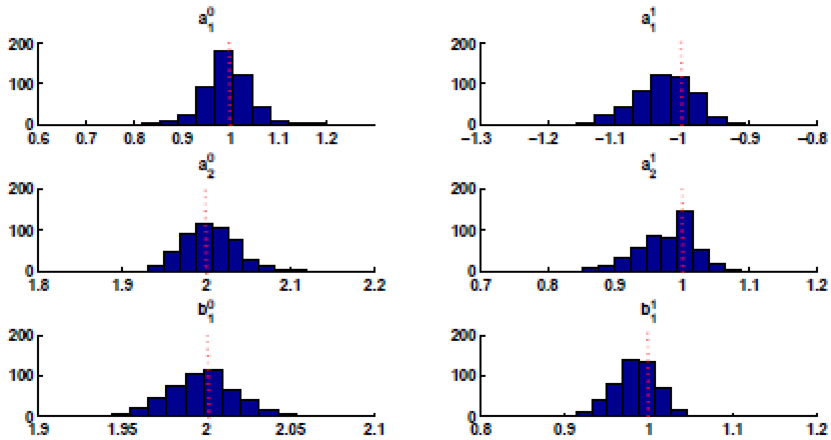


**Figure 5: Distribution of *fLPV-SRIV* estimates *SNR* = 10 dB.**

**5.1.4. CHOICE OF THE FRACTIONAL ORDER**

The $l_2$-norm (in dB) of the normalized output error is defined by:

$$J_{dB} = 10\log\left(\frac{\|y-\hat{y}\|^2}{\|y\|^2}\right) \tag{54}$$

where $y$ and $\hat{y}$ are, respectively, the measured and the estimated output.

The $l_2$-norm (in dB) is evaluated for different values of the commensurate order $\upsilon \in [0,1.2[$ and plotted in Figure 6. By applying the *fLPV-SRIV* algorithm, the optimal value of $J_{dB}$ is found at $\upsilon = 0.75$ and the criterion at the optimum is close to $-SNR$ ( $J_{dB} \simeq -20$ dB).

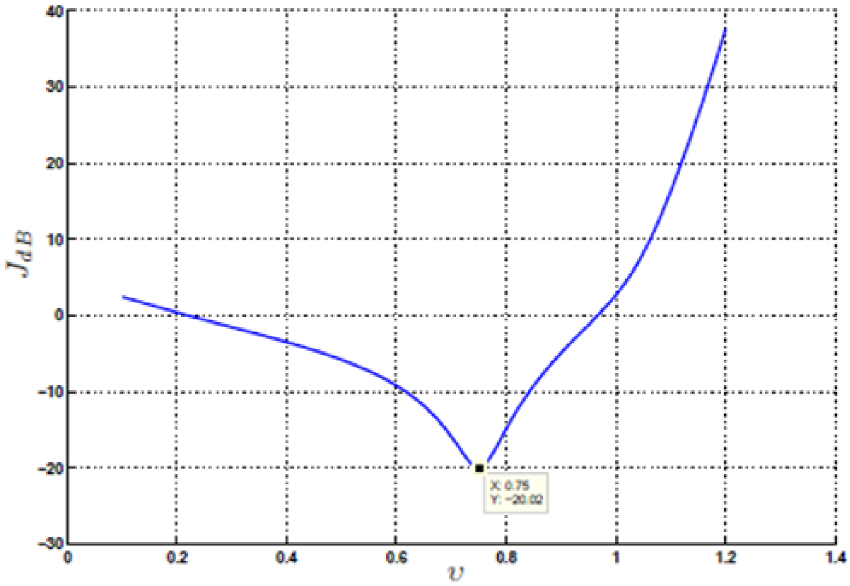



**Figure 6: Criterion versus commensurate fractional order.**

### *5.2.* ESTIMATION OF THE COMMENSURATE ORDER

In this section, assuming an unknown commensurate order. The model structure is set to equation (49), with an unknown commensurate order $\upsilon$.
The initial commensurate order is set to be integer $\upsilon^0 = 1$. $N_{mc} = 300$ runs of Monte Carlo simulation are considered with $SNR = 20$ dB. The estimation results are summarized in table 3.

**Table 3: Estimation of the commensurate order.**

| Parameter | Mean | std | NRQE |
|---|---|---|---|
| $a_1^0 = 1$ | 1.0309 | 0.0607 | |
| $a_1^1 = -1$ | -1.0070 | 0.0813 | |
| $a_2^0 = 2$ | 1.9942 | 0.0208 | 0.0267 |
| $a_2^0 = 1$ | 1.0044 | 0.0266 | |
| $b_0^0 = 2$ | 2.0122 | 0.0391 | |
| $b_0^1 = 1$ | 1.0026 | 0.0483 | |
| $\upsilon = 0.75$ | 0.7522 | - | |

**Figure 7 presents the histograms of the *OO-fLPV-SRIV* estimates.**

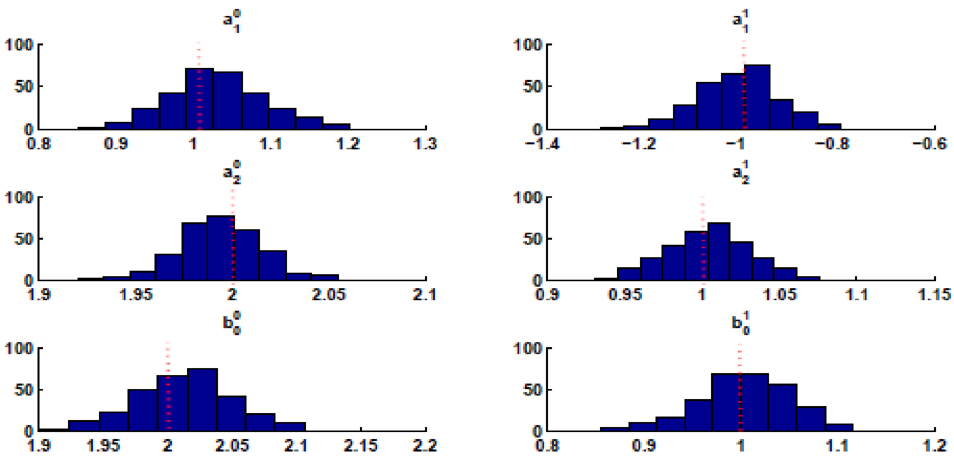



**Figure 7: Distribution of *OO-fLPV-SRIV* estimates**

All the estimates converge to the true ones with a minimum value of *NRQE*

## VI CONCLUSION

In this paper, three new methods are developed to deal with the fractional LPV system identification using a direct approach. All the estimators are based on a global approach using the input, the output and the scheduling variable signals. In the case of unknown coefficients and differentiation orders, a nonlinear optimization algorithm is combined with the optimal instrumental variable method to estimate both of them. The efficiency of the proposed estimators is analyzed through a numerical example via Monte Carlo simulations and under given conditions. Results have shown that the *fLPV-SRIV* is the best estimator. Its performances have demonstrated that it is robust to noise. An interesting perspective is the extension of the proposed methods for error in variable context.

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