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Exact solution of position dependent mass Schrödinger equation for the Hulthen plus Coulomb tensor potential

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ABSTRACT
One of the main objectives in theoretical physics since the early years of quantum mechanics (QM) is to obtain an exact solution of the Schrödinger equation for some special physically important potentials. Since the wave function contains all necessary information for full description of a quantum system. Analytical solution of the Schrödinger equation is of high importance in non-relativistic and relativistic quantum mechanics. The development of potential theory offers compelling coarse-grained descriptions of fundamental interactions in quantum field theory We have obtained exact solution of the position dependent mass Schrödinger equation for the Hulthen plus Coulomb tensor potential. Using the Nikiforov–Uvarov method, we analytically develop the relativistic energy eigenvalues of wave functions. The exact bound state energy eigenvalues and corresponding eigenfunctions are presented. The bound state eigenfunctions are obtained in terms of the hypergeometric functions.
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I. I. Introduction

One of the fundamental methods used to obtain the exact solutions of quantum systems is called the series expansion method, which has been widely applied to study some quantum problems, particularly in obtaining the exact solutions of the hydrogen atom in almost all of classical textbooks[1]. A position dependent effective mass, $M(y) = m_0 m(y)$ associated witha quantum mechanical particle constitutes a useful model for the study of various potentials [4-11]. Systems with certain types of effective masses are found to be useful in the determination of physical properties of semiconductor heterostructures, quantum liquids and dots, helium and metal type clusters, and nuclei as well [13]. Quantum mechanical systems with a position dependent mass (PDM) generate interest for its relevance and importance in describing the physics of many microstructures of current interest, understanding transport phenomena in compositionally graded crystals, designing modern fabrication of nano devices such as quantum dots, wires and wells, developing theoretical models for effective interactions in nuclear physics, neutron stars, liquid crystals, metal clusters. These applications have stimulated a naturally renewed interest in the solution of PDM quantum mechanical Hamiltonians. Recently, solution of the PDM wave equation, have received much attention [14]. In different works complete classification scheme for kinetic energy operators (KEO's) describing a particle endowed with position dependent mass (PDM) [15].

In the case of external hyperbolic-tangent potential investigated the Schrödinger equation for a particle with a nonuniform solitonic mass density [18] and in extend discussed the (nontrivial) position-dependent mass $V(x) = 0$ case whose solutions are hypergeometric functions in $\tanh^2 x$. By using the PCT method [19] re-examinated a new model of semiconfined harmonic oscillator with a mass that varies with position, which has the striking property of having the same spectrum as the standard harmonic oscillator model. In [20] using a recently developed technique to solve the Schrödinger equation for constant mass, studied the regime in which the mass varies with position, i.e. the position-dependent mass Schrödinger equation. A general point canonical transformation is applied for solution of Schrödinger equation for the Scarf and generalized harmonic oscillator potentials with the position-dependent mass using a free parameter [21]. For exact solutions of the position-dependent-effective mass Schrödinger equation several techniques were developed [22] and shown that the exact solutions are limited to a small set of systems. The objective of this paper is to investigate the position dependent effective mass Schrodinger equation for the Hulten plus Coulomb tensor potential by using the Nikiforov-Uvarov (NU) method.

II. Position-Dependent Mass Schrödinger Equation

The one dimensional time dependent Schrödinger equation (SE) within the case of spatially dependent mass is written

$$-\frac{^2}{2}\left[\nabla_y \frac{1}{M(y)}\nabla_y\right]\Psi(y)-\left[E-V_{eff}(y)\right]\Psi(y)=0 \tag{1}$$

Here the mass function $M(y)=m_0 m(y)$. Primes stand for the derivatives with respect to y and we have set $=2m_0=1$.Hence, the SE takes the form

$$\left(-\frac{1}{m(y)}\cdot\frac{d^2}{dy^2}+\frac{m'(y)}{m^2}\cdot\frac{d}{dy}+V_{eff}-E\right)\Psi(y)=0 \tag{2}$$

$$\left(\frac{d^2}{dy^2}-\frac{m'(y)}{m(y)}\cdot\frac{d}{dy}+m(y)\left(E-V_{eff}\right)\right)\Psi(y)=0 \tag{3}$$

where V_{eff} has the form

$$V_{eff}=V_H+V_C \tag{4}$$

The potential consists of the Hulthén

$$V_H(y)=-\frac{Ze^2\delta e^{-\delta y}}{1-e^{-\delta y}} \tag{5}$$

plus Coulomb tensor potential

$$V_C(y)=-\frac{Q}{r},\ Q=\frac{Z_aZ_b}{4\pi\varepsilon_0},\ r\geq R. \tag{6}$$

The parameter *Z* is the atomic number, while *δ* and *y* are the screening parameter and separation distance of the potential, respectively. The parameters *a* and *b* indicate the interaction strengths. The Hulthen potential is one of the important short-range potentials in physics. The potential has been used in nuclear and particle physics, atomic physics, solid-state physics, and its bound state and scattering properties have been investigated by a variety of techniques. General wave functions of this potential have been used in solid-state and atomic physics problems. It should be noted that, Hulth´en potential is a special case of Eckart potential [2-3]. We use the following improved approximation for $\delta y \quad 1$ [17].

$$\frac{1}{y}\approx\frac{\delta}{(1-e^{-\delta y})},\qquad \frac{1}{y^2}\approx\frac{\delta^2}{(1-e^{-\delta y})^2} \tag{7}$$

It provides a good accuracy for a small value of potential parameters. Here, we consider the following mass distribution:

$$m(y)=(1-e^{-\delta y})^{-1} \tag{8}$$

Explicitly, from the two mentioned potentials, we have

$$V(r)=-\frac{Ze^2\delta e^{-\delta y}}{1-e^{-\delta y}}-\frac{Z_aZ_b}{4\pi\varepsilon_0}\cdot\frac{\delta e^{-\delta y}}{1-e^{-\delta y}}=-\frac{(A+B)e^{-\delta y}}{1-e^{-\delta y}} \tag{10}$$

with $A=Ze^2\delta$, $B=\frac{Z_aZ_b}{4\pi\varepsilon_0}\delta$. Having inserted the Eq. (10) into Eq.(1) we obtain

$$\frac{d^2\Psi(y)}{dy^2}-\frac{m'(y)}{m(y)}\cdot\frac{d\Psi(y)}{dy}+m(y)\left(E+\frac{(A+B)e^{-\delta y}}{1-e^{-\delta y}}\right)\Psi(y)=0 \tag{11}$$

The above equation can be further simplified using a new variable $z=e^{-\delta y}$, and $0\leq z\leq 1$, it will obtained as

$$\frac{d^2\Psi(z)}{dz^2}+\frac{1-2z}{z(1-z)}\cdot\frac{d\Psi(z)}{dz}+\frac{1}{(1-z)^2z^2}\left[-\varepsilon^2(1-z)+\alpha^2 z(1-z)\right]\Psi(z)=0 \tag{12}$$

where

$$\begin{cases} \varepsilon^2 = -\dfrac{1}{\delta^2}\cdot E \\ \alpha^2 = \dfrac{(A+B)}{\delta^2} \end{cases} \tag{13}$$

III. Methodology

Now for implementing NU method, Eq.(12) should be rewritten as the hypergeometric type equation form presenting below:

$$\varphi''(z) + \frac{\tilde{\tau}(z)}{\sigma(z)}\varphi'(z) + \frac{\tilde{\sigma}(z)}{\sigma^2(z)}\varphi(z) = 0 \tag{14}$$

After comparing the Eq.(12) and Eq.(14), we obtain

$$\tilde{\tau}(z) = 1-2z\,,\ \sigma(z) = z(1-z)\,,\ \tilde{\sigma}(z) = -\varepsilon^2(1-z) + \alpha^2 z(1-z) \tag{15}$$

The new function in [16] $\pi(z)$ can be found by substituting Eq.(15) and taking $\sigma'(z) = 1-2z$. Hence, the function $\pi(z)$

$$\pi(z) = \pm\sqrt{(a-k)z^2 - (b-k)z + c} \tag{16}$$

where,

$$a = \alpha^2,\ b = \varepsilon^2 + \alpha^2,\ c = \varepsilon^2 \tag{17}$$

The constant parameter *k* can be found performing the condition that the discriminant of the expression Eq.(16) under the square root is equal to zero. Hence, we obtain:

$$k_{1,2} = (b-2c) \pm 2\sqrt{c^2 + c(a-b)} \tag{18}$$

When the individual values of are given in Eq. (18) are substituted into Eq.(16), the eight possible forms of $\pi(z)$ are written in the following forms:

$$\pi(z) = \pm\begin{cases} \left(\sqrt{c} - \sqrt{c+a-b}\right)z - \sqrt{c}, & for\ \ k = (b-2c) + 2\sqrt{c^2 + c(a-b)} \\ \left(\sqrt{c} + \sqrt{c+a-b}\right)z - \sqrt{c}, & for\ \ k = (b-2c) - 2\sqrt{c^2 + c(a-b)} \end{cases} \tag{19}$$

The polynomial $\pi(z)$ have four possible forms according to NU method, but we select the one of them which the functions of $\tau(z) = \tilde{\tau}(z) + 2\pi(z)$ [16] has the negative derivative, i.e. $\tau'(z) = -\left[2 + 2(\sqrt{c} + \sqrt{c+a-b})\right]\ 0$ and $\tau(z) = 0$ in the $z \in (0,1)$. Another forms have not physically meaning. Therefore, the appropriate $\pi(z)$

$$\pi(z) = \sqrt{c} - z\left[\sqrt{c} + \sqrt{c+a-b}\right] \tag{20}$$

and

$$\tau(z) = 1 - \left[2 + 2(\sqrt{c} + \sqrt{c+a-b})\right]z + 2\sqrt{c} \tag{21}$$

$$k = (b-2c) - 2\sqrt{c^2 + c(a-b)} \tag{22}$$

$$\lambda = k + \pi'(z) \tag{23}$$

To find a physical solution, the expression in the square root must be square of a polynomial. Then, a new eigenvalue equation for the SE becomes

$$\lambda_n = -n\tau'(z) - \frac{n(n-1)}{2}\sigma''(z) \tag{24}$$

The following expression for the constant λ are obtained, respectively

$$\lambda = (b-2c) - 2\sqrt{c^2 + c(a-b)} - \left[\sqrt{c} + \sqrt{c+a-b}\right] \tag{25}$$

$$\lambda_n = -n\tau'(z) - \frac{n(n-1)}{2}\sigma''(z) = n\left[2 + 2\sqrt{c} + 2\sqrt{c+a-b}\right] + n(n-1) \tag{26}$$

$$(b-2c) - 2\sqrt{c^2 + c(a-b)} - \left[\sqrt{c} + \sqrt{c+a-b}\right] = 2n\left[1 + \sqrt{c} + \sqrt{c+a-b}\right] + n(n-1) \tag{27}$$

If we solve equation (27) for $\sqrt{c}$, we find analytical formul:

$$\sqrt{c} = \frac{b - 2c - (2n+1)\sqrt{c+a-b} - n(n+1)}{1 + 2n + 2\sqrt{c+a-b}} \tag{27}$$

The resulting energy eigenvalue is

$$E_n = -\left[\frac{\sqrt{(1+2n)^2 + 8(b - n(n+1))} - (2n+1)}{4} \cdot \delta\right]^2 \tag{28}$$

By applying NU method, we can obtain the radial eigenfunctions. After substituting and into $\frac{\phi'(z)}{\phi(z)} = \frac{\pi(z)}{\sigma(z)}$ and [12] solving the order differential equation, one can find the finite function $\phi(z)$ and in the interval it is easily obtained

$$\Psi(z) = \phi(z) y(z) \tag{29}$$

$$\frac{\phi'(z)}{\phi(z)} = \frac{\pi(z)}{\sigma(z)}$$

$$\phi(z) = z^{\varepsilon} \tag{30}$$

Beyond that, the other part of the wave function $y_n(z)$ is the hypergeometric type function whose polynomial solutions are given by Rodrigues relation

$$y_n(z) = \frac{C_n}{\rho(z)} \cdot \frac{d^n}{dz^n}\left[\sigma^n(z) \cdot \rho(z)\right] \tag{31}$$

where C_n is the normalization constant and its value is $C_n = \frac{(-1)^n}{2^n n!}$ [23]. After using the following definition of the Jacobi polynomials [23]

$$P_n^{(p,q)}(2z) = \frac{(-1)^n}{n! 2^n (1-z)^p (1+z)^q} \frac{d^n}{dz^n}\left[(1-z)^{p+n}(1+z)^{q+n}\right] \tag{32}$$

the Eq.(31) passes to the following form:

$$y_n(z) = C_n \cdot P_n^{(2\varepsilon)}(1-2z) \tag{33}$$

Using the experssions (30) and (31) and putting them into experssion (29), for radial wave function $\Psi(z)$ we find

$$\Psi_n(z) = C_n \cdot z^{\varepsilon} \cdot \frac{\Gamma(n+2\varepsilon+1)}{n!\Gamma(2\varepsilon+1)} {}_2F_1\left(-n, 2\varepsilon+n, 2\varepsilon+1; z\right)$$

It's known that C_n - is normalizing constant and is found from normalization condition:

$$\int_0^\infty |R(r)|^2 r^2 dr = \int_0^\infty |\Psi(r)|^2 dr = 2\delta \int_0^\infty \frac{1}{z} |\Psi(z)|^2 dz = 1 . \tag{34}$$

For this we use the integral formula

$$\int_0^1 (1-2z)^{2(\delta+1)} \cdot z^{2\lambda-1}\left\{{}_2F_1(-n, 2(\delta+\lambda+1)+n, 2\lambda+1; z)\right\}^2 dz =$$

$$= \frac{(n+\delta+1)n!\Gamma(n+2\delta+2)\Gamma(2\lambda)\Gamma(2\lambda+1)}{(n+\delta+\lambda+1)\Gamma(n+2\lambda+1)\Gamma(2(\delta+\lambda+1)+n)} . \tag{35}$$

$$C_n^2\left(\frac{\Gamma(n+2\varepsilon+1)}{n!\Gamma(2\varepsilon+1)}\right)^2 \cdot \frac{1}{2\delta} \cdot \frac{n \cdot n!\Gamma(n)\Gamma(2\varepsilon)\Gamma(2\varepsilon+1)}{(n+\varepsilon)\Gamma(n+2\varepsilon+1)\Gamma(2\varepsilon+n)} = 1 \tag{36}$$

Then we obtain the normalization constant of the wave function as

$$C_n = \sqrt{\frac{2\delta n!(n+\varepsilon)\Gamma(2\varepsilon+1)\Gamma(2\varepsilon+n)}{n\cdot\Gamma(n)\Gamma(2\varepsilon)\Gamma(n+2\varepsilon+1)}} \quad (37)$$

IV. Conclusion

In this paper, the solutions of the position-dependent effective mass Schrödinger equation for the Hulth´en plus Coulomb tensor potential have been investigated by Nikiforov-Uvarov method. It has been shown that, both the wavefunctions and the corresponding energy spectra of the system have an exact . An appropriate mass function has been introduced for solving the position-dependent effective mass Schrödinger equation.

References

[1]. **Shi-Hai Dong, Wen-Hua Huang, Parisa Sedaghatnia, Hassan Hassanabadi,** Exact solutions of an exponential type position dependent mass problem, Results in Physics, Volume 34 (2022) 105294, https://doi.org/10.1016/j.rinp.2022.105294

[2]. **Hulthen L.,** Uber die eigenl osungender Schrodinger-Gleichung des deuterons, Ark. Mat. Astron. Fys. A, Vol.28, No.5, 1942, pp.1-12.

[3]. **Hulthen L.,** On the virtual state of the deuteren, Ark. Mat. Astron. Fys. B, Vol.29, No.1, 1942, pp.1-11.

[4]. **Ramazan S., Cevdet T,·Özlem Y.¸Mahmut B**. Exact Solution of Effective Mass Schrödinger Equation for the Hulthen Potential. Int J Theor Phys (2008) 47: 2243–2248, DOI 10.1007/s10773-008-9656-7.

[5]. **Sanjib Meyur.** Position Dependent Mass for the Hulthen plusHyperbolic Cotangent Potential. Bulg. J. Phys. **38** (2011) 357–363. http://www.bjp-bg.com/papers/bjp2011_4_357-363.pdf.

[6]. **Ramazan Sever, Cevdet Tezcan** . Exact solution of Schrödinger equation for modified Kratzer's molecular potential with the position-dependent mass. Int. J. Mod. Phys. E 17, 1327(2008). https://doi.org/10.1142/S0218301308010428.

[7]. **Hangshadhar Rajbongshi, Ngangkham Nimai Singh**. Generation of new exactly solvable potentials of position-dependent mass Schrödinger equation by extended transformation method. ACTA PHYSICA POLONICA B, Vol. 45 (2014), No 8. DOI:10.5506/APhysPolB.45.1701.

[8]. **A. Kharab, H. Eleuch.** Analytical solution of the position dependent mass Schrödinger equation with a hyperbolic tangent potential. Int. J of Applied Mathematics. Volume 32 No. 2 2019, 357-367, doi: http://dx.doi.org/10.12732/ijam.v32i2.14.

[9]. **Jiang Yu, Shi-Hai Dong, Guo-Hua Sun**. Series solutions of the Schrödinger equation with position-dependent mass for the Morse potential, Physics Letters A 322 (2004) 290–297, https://doi.org/10.1016/j.physleta.2004.01.039.

[10]. **H. R. Christiansen and M. S. Cunha**, Solutions to position-dependent mass quantum mechanics for a new class of hyperbolic potentials, J. of Mathematical Physics, December 2013, DOI: 10.1063/1.4840615 Source: arXiv.

[11]. **Sanjib Meyur, Smarajit Maji, and S. Debnath**, Analytical Solution of the Schrödinger Equation with Spatially Varying Effective Mass for Generalised Hylleraas Potential, Advances in High Energy Physics, Volume 2014, Article ID 952597, https://doi.org/10.1155/2014/952597.

[12]. **H. I. Ahmadov, E. A. Dadashov, N. Sh. Huseynova, V. H. Badalov**, Generalized tanh-shaped hyperbolic potential: bound state solution of Schrödinger equation, Eur. Phys. J. Plus (2021) 136:244, https://doi.org/10.1140/epjp/s13360-021-01202-8

[13]. **Metin A., Ramazan** S., Effective Mass Schrödinger Equation for Exactly Solvable Class of One-Dimensional Potentials, J. Mathematical Chemistry, Volume 43, pages 92-100 (2008), DOI:10.1007/s10910-006-9181-y

[14]. **Ramazan K., Seda S**., Remarks on the Solution of the Position Dependent Mass (PDM) Schrödinger Equation, https://doi.org/10.1088/1751-8113/43/45/455203.

[15]. **A. Trabelsi, F. Madouri, A. Merdaci and A. Almatar**., Classification Scheme for Kinetic Energy Operators with Position-Dependent Mass, https://doi.org/10.48550/arXiv.1302.3963.

[16]. A.F. Nikiforov, V.B. Uvarov, Special Functions of Mathematical Physics (Birkhäuser Basel, 1988), https://doi.org/10.1007/978-1-4757-1595-8

[17]. **Ahmet Tas, Oktay Aydogdu and Mustafa Salti,** Relativistic Spinless Particles with Position Dependent Mass: Bound States and Scattering Phase Shifts. Journal of the Korean Physical Society, Vol. 70, No. 10, May 2017, pp. 896~904

[18]. **M. S. Cunha and H. R. Christiansen**, Analytic results in the position-dependent mass Schrodinger problem, Commun. Theor. Phys. vol. 60, No. 6, pp. 642-650, 2013, https://doi.org/10.48550/arXiv.1306.0933.

[19]. **C. Quesne**, Generalized semiconfined harmonic oscillator model with a position-dependent effective mass, Eur. Phys. J. Plus (2022) 137:225, https://doi.org/10.48550/arXiv.2110.10581.

[20]. **Pankaj K. Jhaa, Hichem Eleuchb and Yuri V. Rostovtsev**, Analytical solution to position dependent mass Schrödinger equation, Journal of Modern Optics, 2011, 1–5, iFirs, DOI: 10.1080/09500340.2011.562617.

[21]. **Cevdet Tezcan, Ramazan Sever,** PT-Symmetric Solutions of Schrödinger Equation with Position-Dependent Mass via Point Canonical Transformation, Int J Theor Phys (2008) 47: 1471–1478, DOI 10.1007/s10773-007-9589-6

[22]. **M. Sebawe Abdalla and H. Eleuch,** Exact solutions of the position-dependent-effective mass Schrödinger equation, AIP ADVANCES 6, 055011 (2016), http://dx.doi.org/10.1063/1.4949567

[23]. **M. Abramowitz, I.A. Stegun**, Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables (Dover, New York, 1964)