The International Journal of Engineering and Science (IJES)
|| Volume || 13 || Issue || 11 || Pages || PP 84-91 || 2024 ||
ISSN (e): 2319 – 1813 ISSN (p): 2319 – 1805

Topological Isomorphism between Certain Algebras on the Euclidean motion group.

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ABSTRACT

Given $G = \mathbb{R}^n \rtimes SO(n)$ and $K := SO(n)$, its compact subgroup. The set of spherical functions on G that are bounded, denoted as Σ, is considered as the Gelfand spectrum of G. Among other notable results, a topological isomorphism is established between the Banach algebra of K-bi-invariant functions $(L^1(K\backslash G/K))$ on G and Σ.

Date of Submission: 09-11-2024 Date of acceptance: 23-11-2024

Introduction

Some of the special functions introduced in analysis are related to the representations of Lie groups ([4]). Prominent among such functions are the spherical functions. The theory of spherical function generalizes the classical Laplace spherical harmonics and continuous characters of Lie groups. Spherical functions are significant in the modern theory of infinite dimensional linear representation of Lie groups. In this work we discuss spherical function on the Euclidean motion groups. Adopting the realization of the set of bounded positive definite spherical functions on the pair $(\mathbb{R}^n \rtimes SO(n), SO(n))$ as the Gelfand spectrum Σ, we prove, among other things, that there is a topological isomorphism between the commutative convolution algebra $L^1(K\backslash G/K)$ and the Gelfand spectrum. This study is organized into three sections. Sections two deals with the structure of motion group and its irreducible unitary representation. In section three, main results concerning this study are presented.

2. Preliminaries

2.1 The Euclidean Motion Groups. The group $SE(n)$, known as the Euclidean motion group, is realized as the semi-direct product of $\mathbb{R}^n$ with $SO(n)$. That is $SE(n) = \mathbb{R}^n \rtimes SO(n)$. A member of $SE(n)$ may be denoted as $g = (\bar{x}, \xi)$, where $\xi \in SO(n)$ and $\bar{x} \in \mathbb{R}^n$. For any $g_1 = (\bar{x_1}, \xi_1)$ and $g_2 = (\bar{x_2}, \xi_2) \in SE(n)$, multiplication on SE(n) may be defined as

$$g_1 g_2 = (\bar{x_1} + \xi_1 \bar{x_2}, \xi_1 \xi_2),$$

and the inverse is defined as

$$g^{-1} = (-\xi^t \bar{x}, \xi^t).$$

Here ξ^t denotes a transpose. Alternatively, $SE(n)$ may also be identified with a matrix group whose arbitrary element may be identified as $(n+1) \times (n+1)$ matrix given by

$$H(g) = \begin{pmatrix} \xi & \bar{x} \\ 0^t & 1 \end{pmatrix},$$

where $\xi \in SO(n)$ and $0^t = (0, 0..., 0)$. It is observed that $H(g_1)H(g_2) = H(g_1 g_2)$, $H(g^{-1}) = H^{-1}(g)$ and $g \mapsto H(g)$ an isomorphism between $SE(n)$ and H(g).

The matrix representation of the element of $SE(2) \subset GL(3, \mathbb{R})$ is given as

$$g((x_1, x_2), \phi) = \begin{pmatrix} cos\phi & -sin\phi & x_1 \\ sin\phi & cos\phi & x_2 \\ 0 & 0 & 1 \end{pmatrix},$$

where $\phi \in [0, 2\pi]$, $(x_1, 2_2) \in \mathbb{R}^2$ ([7],[11]) and in polar coordinate as

$$g(\bar{x}, \phi, \theta) = \begin{pmatrix} cos\phi & -sin\phi & x_1 cos\theta \\ sin\phi & cos\phi & x_2 sin\theta \\ 0 & 0 & 1 \end{pmatrix},$$

$\bar{x} = (x_1, x_2) \in \mathbb{R}^2$. $\phi, \theta \in [0, 2\pi]$. The group $SE(2)$ is a non - compact and non- commutative solvable Lie group ([3]). $\forall n \geq 2$, $SE(n)$ is a group of affine maps induced by orthogonal transformations. It is also referred to as a group of rigid motions on $\mathbb{R}^n$ and plays a significant role in robotic, motion planning as well as dynamics ([7],[2]). The group action of $M(2)$ is rotation operation that is followed by translation on the plane. That is g translate

$$(x_1, x_2)^T to\ (x_1', x_2')^T$$

as follows

$$x_1' = x_1 cos\phi - x_2 sin\phi + a_1$$

$$x_2' = x_1 sin\phi + x_2 cos\phi + a_2$$

and $(x_1', x_2') = g.(x_1, x_2)$([7], p.3) $SE(2)$ is also called the Isometry group of $\mathbb{R}^2$, which is sometimes denoted as $I(\mathbb{R})^2$.

2.2 Irreducible Unitary representation of SE(2). A comprehensive description of the representation of $SE(2)$ is given in this section. It is also shown that this representation is irreducible and unitary. Let $K = SO(2)$, a compact subgroup of G, and let $L^2([0, 2\pi], \frac{d\alpha}{2\pi})$ be the Hilbert space on $\mathbb{T} \cong [0, 2\pi] \cong SO(2)$. A representation of $SE(2)$ on $L^2(K)$ is an operator defined by

$$U(g, p)\tilde{\varphi}(X) = e^{-ip(x.X)}\tilde{\varphi}(\xi^T X)$$

for each $g = (x, \xi) = g((x_1, x_2), \xi) \in SE(2)$, $p \in \mathbb{R}^+$, $X.y = x_1y_1 + x_2y_2$, X is a unit vector ([6]). $U(g, p)$ is unitary and irreducible. Following the approach of Vilenkin ([7], p. 200 and [8]), a representation of $SE(n)$ on $L^2(K)$ is defined as

$$T_R(g)f(x) = e^{R(a,x)}f(x_{-\alpha}) \tag{1}$$

where $x_{-\alpha}$ is the vector into which x is transformed under a rotation by an angle $-\alpha$ and $(a, x) = a_1x_1 + a_2x_2$. Following (1), we show that $T_R(g)$ is irreducible and unitary in what follows. The parametric equation of the circle $x_1^2 + x_2^2 = 1$ has the form

$$x_1 = cos\psi$$

$$x_2 = sin\psi$$

$0 \leq \psi \leq 2\pi$. Therefore, one can regard functions $f(x)$ on the space ,$\mathfrak{D}$, of square integrable functions on $K = SO(2)$ as functions of ψ. That is

$$f(x) \equiv f(\psi).$$

The operator $T_R(g)f(x)$ can then be written as

$$T_R(g)f(\psi) = e^{Rrcos(\psi-\varphi)}f(\psi - \alpha),$$

where

$$a = (rcos\phi, rsin\phi),\ g = g(a, \alpha).$$

Let

$$(f_1, f_2) = \frac{1}{2\pi}\int_0^{2\pi} f_1(\psi)\overline{f_2(\psi)}d\psi$$

be a scalar product defined in $\mathfrak{D}$. Completing $\mathfrak{D}$ with respect to this product produces a Hilbert space denoted by $\mathcal{H}$. If $R = i\rho$, $T_R(g)$ is unitary. Since $T_R(g)$ is a faithful representation, it means that $T_R(g_1) \neq T_R(g_2)$ if $g_1 \neq g_2$. The irreducibility of $T_R(g)f(x)$ is presented as follows. The Lie algebra $\mathfrak{se}(2)$ of $SE(2)$ has the following basis

$$X_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, X_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} X_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

X_1, X_2, and X_3 satisfy the following commutation relations $[X_1, X_2] = 0$, $[X_2, X_3] = X_1$ and $[X_3, X_1] = X_2$ and their corresponding one - parameter subgroups are (see [1]) for details.

$$g_1(t) = \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, g_2(t) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}, g_3(t) = \begin{pmatrix} cost & -sint & 0 \\ sint & cost & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

respectively. The operator $T_R(g_1(t))$ transforms the function $f(\psi)$ into

$$T_R(g_1(t)) \ = e^{Rtcos\psi} f(\psi).$$

Let us define

$$A_1 = \frac{d}{dt} T_R(g_1(t))\Big|_{t=0} = \frac{d}{dt} e^{Rtcos\psi}\Big|_{t=0} = Rcos\psi e^{Rtcos\psi}\Big|_{t=0} = Rcos\psi$$

This shows that A_1 is the operator of multiplication by $Rcos\psi$.
We also define

$$A_2 = \frac{d}{dt} T_R(g_2(t))\Big|_{t=0} = \frac{d}{dt} e^{Rtsin\psi}\Big|_{t=0} = Rsin\psi e^{Rtsin\psi}\Big|_{t=0} = Rsin\psi$$

Lastly,

$$A_3 = \frac{d}{dt} T_R(g_3(t))\Big|_{t=0} = \frac{d}{dt} f(\psi - t) = -\frac{d}{dt}$$

The operators A_1, A_2 and A_3 also satisfy commutation relations as follows.

$$[A_1, A_2] = A_1A_2 - A_2A_1 = Rcos\psi Rsin\psi - Rsin\psi Rcos\psi = 0$$

that is,

$$\Big[A_1, A_2\Big] = 0$$

Also,

$$[A_2, A_3] = Rsin\psi(\frac{-d}{d\psi}) - (\frac{-d}{d\psi})(Rsin\psi)$$
$$= -Rsin\psi\frac{d}{d\psi} + \frac{d}{d\psi} Rsin\psi$$
$$= -Rsin\psi\frac{d}{d\psi} + Rcos\psi = Rcos\psi,$$

implying that

$$\Big[A_2, A_3\Big] = A_1.$$

Finally,

$$[A_3, A_1] = -\frac{d}{d\psi} Rcos\psi - (Rcos\psi(-\frac{d}{d\psi})$$
$$-\frac{d}{d\psi} Rcos\psi + (Rcos\psi(\frac{d}{d\psi})$$
$$= Rsin\psi + 0 = Rsin\psi,$$

showing that

$$\Big[A_3, A_1\Big] = 0$$

Next, it is shown that for $R \neq 0$, the representations $T_R(g)$ are irreducible. It suffices to prove that any non - trivial stable subspace in $\mathcal{H}$ ($\mathcal{H} = L^2(T)$) coincides with $\mathcal{H}$. In order to achieve this, T_R is restricted onto $SO(2)$. Let the following notations be introduced. $A_1 e^{ik\psi} = Rcos\psi e^{ik\psi}$, $A_2 e^{ik\psi} = Rsin\psi e^{ik\psi}$, $A_3 e^{ik\psi} = -ike^{ik\psi}$. With respect to this notations, the following linear combinations hold. $H_+ = A_1 + iA_2 = Re^{i\psi}$, $H_- = A_1 - iA_2 = Re^{-i\psi}$. Now, $H_+ e^{ik\psi} = Re^{i\psi} e^{ik\psi} = Re^{i(1+k)\psi}$ and $H_- e^{ik\psi} = Re^{-i\psi} e^{ik\psi} = Re^{i(1-k)\psi}$. Therefore,

$$\begin{aligned}[H_+, H_-] &= H_+H_- - H_-H_+ \\ &= Re^{i\psi} Re^{-i\psi} - Re^{-i\psi} Re^{i\psi} \\ &= 0,\end{aligned}$$

$$\begin{aligned}[H_+, A_3] &= H_+A_3 - A_3H_+ \\ &= Re^{i\psi}(\frac{-d}{d\psi}) - [-\frac{-d}{d\psi}Re^{i\psi}] \\ &= 0 + \frac{d}{d\psi}Re^{i\psi} \\ &= iRe^{i\psi} \\ &= iH_+\end{aligned}$$

$$\begin{aligned}[H_-, A_3] &= H_-A_3 - A_3H_- \\ &= Re^{-i\psi}(\frac{-d}{d\psi}) - (\frac{-d}{d\psi})Re^{-i\psi} \\ &= 0 + \frac{d}{d\psi}Re^{-i\psi} \\ &= -iRe^{-i\psi} \\ &= iH_-\end{aligned}$$

The restriction of T_R onto $SO(2)$ is the regular representation of $SO(2)$ and could be decomposed into the direct sum of one dimensional representation realized in $\mathcal{H}_K$, being a subspace of functions that takes the form $C_k e^{ik\psi}$. If J is defined to be an arbitrary subspace that is invariant with respect to $SE(2)$ and $SO(2)$, it could be decomposed into a sum of $\mathcal{H}_K$, which are also subspaces, and consequently, it is either the set is null or it contains one of the functions $e^{ik\psi}$. The invariance of J means that, along with any one of $e^{ik\psi}$, it contains all the functions $H_+^m e^{ik\psi}$ and $H_-^m e^{ik\psi}$, therefore, it contains all subspaces $\mathcal{H}_K$. That is, J coincides with $\mathfrak{D}$. So, T_R, $R \neq 0$, is irreducible. If R is identically zero, then T_R takes the form

$$(T_0(g)f)(\psi) = f(\psi - \alpha),\ g = (a, \alpha),$$

and could be decomposed into the direct sum of one - dimensional representation of the form,

$$T_{0n}(g) = e^{in\alpha}.$$

Therefore, the representations T_R, $R \neq 0$, and T_{0n}, $n \in \mathbb{Z}$ exhaust all irreducible representations of $SE(2)$.

2.3 Spherical functions and transforms on SE(2)

2.3.1 Definitions. Let G be a locally compact group and let K be a compact subgroup of G and let $L^1(G)$ be the convolution algebra of absolutely integrable functions on G. A function $f : G \to \mathbb{C}$ is said to be K-bi-invariant if it is constant in the double coset of K, that is,

$$f(k_1 g k_2) = f(g),\ \forall k_1, k_2 \in K \ and\ g \in G.$$

Let $C_c(G)$ denote the space of continuous functions on G with compact support. Then $C_c(G)^K$ denotes the space of K-bi-invariant functions on $C_c(G)$ and, similarly, $L^1(G)^K$ denotes the space of K-bi-invariant functions on $L^1(G)$. The pair (G, K) is a Gelfand pair if $L^1(G)^K$ is a commutative algebra.

We need definitions of spherical functions and transforms of K-bi-invariant functions on G. First, we present the definition of spherical functions.

2.3.2 Definition. A spherical function

$$\varphi : G \to \mathbb{C}$$

for the Gelfand pair (G,K) is a K-bi-invariant C^∞ - function on K with $\varphi(e) = 1$, where e is the identity element of G, and satisfies one of the following equivalent conditions

1. $\int_K \varphi(xky)d\mu_K(k) = \varphi(x)\varphi(y)$, $x, y \in G$, $k \in K$;
2. $f \to \int_G f(g)\overline{\varphi(g)}dg$ is a homomorphism of $C_c(K\backslash G/K)$ into $\mathbb{C}$
3. φ is an eigen function of each $D \in \mathfrak{D}(G/K)$, where $\mathfrak{D}(G/K)$ is the algebra of K-invariant differential operators on G/K(= symmetric space of G).

Also, a function $\varphi \in C(G)$, $\varphi \neq 0$, is said to be spherical if it is bi-invariant under K and χ_φ is a character of $C_c(G)^K$. That is, $\forall f, g \in C_c(G)^K$

$$\chi_\varphi(f * g) = \chi_\varphi(f) \cdot \chi_\varphi(g)$$

Next, we give a definition of spherical transform of a K-bi-invariant function on G. Before going on, some notations that are required are put in place.

Let $S = S(G, K)$ be the set of all spherical functions for the Gelfand pair (G, K), and let $BS(G, K)$ or $(G, K)^+$ denote the subset of $S(G, K)$ consisting of bounded spherical functions (relative to (G,K)). Spherical transform for functions on G may be defined as follows.

2.3.4 Definition. The spherical transform for the Gelfand pair (G, K) is the map

$$f : C_c(K\backslash G/K) \to S(G, K)$$

or

$$f : L^1(K\backslash G/K) \to BS(G, K)$$

defined by

$$\widehat{f}(\varphi) = \int_G f(g)\varphi(g^{-1})d\mu_G(g).$$

The space of bounded spherical spherical function may be topologised by any of the following topologies:

(i) Compact open topology, obtained as the topology of uniform convergence on the compact subset of G.
(ii) The weak topology from the family of continuous linear maps $f : BS(G, K) \to \mathbb{C}$
(iii) The $weak^*$- topology inherited from $L^\infty(K\backslash G/K)$. This is because $BS(G, K) \subset L^\infty(K\backslash G/K)$.

A function $f : G \to \mathbb{C}$ is said to be positive definite if the inequality holds

$$\sum_{i,j=1^m} \alpha_i \overline{\alpha_k} f(g_i^{-1} g_k) \geq 0 \qquad (2)$$

for all subsets $\{g_1, ..., g_m\}$ of elements of G and all sequences $\{\alpha_1, ..., \alpha_m\}$ of complex numbers. The integral analogue of the inequality (2) is given by

$$\int_G \int_G f(g_i^{-1} g_k)\varphi(g_i)\varphi(g_k)dg_i dg_k \geq 0 \qquad (3)$$

where φ ranges over $L^1(G)$ or over the space $C_c(G)$ of continuous functions with compact support. If f is a continuous functions, (2) and (3) are equivalent. A measure π on $(G, K)\widehat{}$ is called the plancherel measure and its support is the set $(G, K)\widehat{}$.

Let Σ represents the set of bounded spherical function. When Σ is endowed with any of the above listed topologies, it is referred to as the Gelfand spectrum of $L^1(K\backslash G/K)$ or the spectrum of (G, K) and the bounded spherical functions defined through the formula

$$f \mapsto \int_G f(g)\varphi(g^{-1})dg$$

determines the multiplicative functional on the corresponding L^1 - algebra. The weak * topology, being one of the topologies on the Gelfand spectrum, is induced from $L^\infty(G)$ and is found to coincide with the compact open topology [5].

3. Main Results

Let us denote $L^1(K\backslash G/K)$ by $\mathcal{A}$ and let X be a locally compact Hausdorff space. Here and hereafter, $C_\infty(X)$ is the space of all continuous functions $\phi : X \to \mathbb{C}$ that vanish at infinity, with norm $\|\phi\|_\infty = Sup_{x\in X}|\phi(x)|$ and $\phi^*(x) = \overline{\phi(x)}$. $C_\infty(X)$ is a commutative C^* - algebra. (That is, it is a Banach * - algebra such that $\|x^*x\| = \|x\|^2$, for all $x \in \mathcal{A}$). The next proposition states clearly that the Gelfand transform is a norm-preserving Banach algebra homomorphism.

3. Proposition[9]

Let $\mathcal{A}$ be a commutative Banach algebra. Then the Gelfand transform $x \mapsto \hat{x}$ is a norm - decreasing Banach algebra homomorphism $\mathcal{G} : \mathcal{A} \mapsto C_\infty(\mathcal{M}_\mathcal{A})$, $\|\hat{x}\|_\infty = \|x\|_{spec} \leq \|x\|_\mathcal{A}$. If $\mathcal{A}$ is a C^* - algebra then $\mathcal{G}$ is a norm preserving * - algebra isomorphism of $\mathcal{A}$ into $C_\infty(\mathcal{M}_\mathcal{A})$

Let $\mathcal{A}^\star$ be the dual of $\mathcal{A}$ and let $D^\star$ be a closed unit disk in $\mathcal{A}^\star$. The next proposition shows that $D^\star$ is compact and Hausdorff.

3.2 Proposition. Let $D^\star$ denote the closed unit disk $\{f \in \mathcal{A}^\star | \parallel f \parallel \leq 1\}$ in $\mathcal{A}^\star$ then $D^\star$ is compact and Hausdorff.
Here $\mathcal{A} = L^1(K\backslash G/K)$ and $\mathcal{A}^* : L^1(K\backslash G/K) \to \mathbb{C}$
Proof. Let $x \in \mathcal{A}$, denote $C_x = \{z \in \mathbb{C} | |z| \leq \|x\|\}$. Then, $C = \prod_{x\in\mathcal{A}} C_x$ is compact in the product topology. Then we map $\mathcal{D}^*$ into $f \to (f(x))$. That is, the x - coordinate of $f \in \mathcal{D}^*$ is $f(x) \in C_x$. This uses the fact that $|f(x)| \leq \|f\| * \|x\|$. The subspace topology on $\mathcal{D}^* \subseteq C$ is the same as the subspace topology on $\mathcal{D}^* \subseteq \mathcal{A}^*$ where C has the product topology and $\mathcal{A}^*$ has the weak $\star$- topology. So $\mathcal{D}^* \subseteq C$. Let h belong to the closure of $\mathcal{D}^*$ in C. In other words, given $\epsilon > 0$ and $x, y \in \mathcal{A}$, there exists $f \in \mathcal{D}^*$ with

$$|h(x) - f(x)| < \frac{\varepsilon}{3}, |h(y) - f(y)| < \frac{\varepsilon}{3} \ and \ |h(x+y) - f(x+y)| < \frac{\varepsilon}{3}$$

then,

$$|h(x+y) - h(x) - h(y)| = |h(x+y) - f(x+y) - h(x) + f(x) - h(y) + f(y)| < \epsilon.$$

This shows that $h(x+y) = h(x) + h(y)\ \forall x, y \in \mathcal{A}$. Similarly, $h(\alpha x) = \alpha h(x)$, for $\alpha \in \mathbb{C}$ and $x \in \mathcal{A}$. This shows that h is linear. If $\epsilon > 0$ and $x \in \mathcal{A}$ there exists $f \in \mathcal{D}^*$ with $|f(x) - h(x)| < \epsilon$, so $|h(x)| < |f(x)| + \epsilon \leq \|x\| + \epsilon$. This shows that $\|h\| \leq 1$. Now $h \in \mathcal{D}^*$. We have proved that $\mathcal{D}^*$ is closed in C. Since C is compact Hausdorff space, $\mathcal{D}^*$ is also compact and Hausdorff.

Let $\mathcal{P}(G, K)$ be the set of positive definite spherical functions on the pair (G,K). It is locally compact in the subspace topology from $BS(G, K)$ or $(G, K)^+$, the subspace topology from $\mathcal{P}(G, K) \subseteq BS(G, K)$ is the same as the subspace topology from $\mathcal{P}(G, K) \subseteq \mathcal{D}^*$ where $\mathcal{D}^*$ is the closed unit disk in the dual space $L^1(K\backslash G/K)^*$, $\mathcal{P}(G, K)$ has a compact closure $cl(\mathcal{P})$ in $\mathcal{D}^*$, and either $cl(\mathcal{P}(G, K)) = \mathcal{P}(G, K)$ or $cl(\mathcal{P}(G, K)) = \mathcal{P}(G, K) \cup \{0\}$.

3.3 Corollary(Riemann-Lebesque Lemma):
If $f \in L^1(K\backslash G/K)$ then $\hat{f}|_{\mathcal{P}} \in (\mathcal{P}(G, K))$.

Let $C^\star(G, K)$ be a Banach algebra that is also a commutative $C^\star$ - algebra and let $\mathcal{R} = \mathcal{R}(G, K)$ be the maximal ideal space $\mathcal{M}_{C^\star(G,K)}$ of $C^\star(G, K)$. The following definitions are in order

3.4 Definition
Let

$$\gamma : L^1(K\backslash G/K) \to C_\infty(R(G, K))$$

be a map defined by $\gamma(f) = \psi\hat{f}$ where $T \mapsto \hat{T}$ is the Gelfand transform $\mathcal{G} : C^\star(G, K) \to C_\infty(R(G, K))$. Then γ is injective and the image is dense in $C_\infty(R(G, K))$. Moreover, if $m \in R = R(G, K)$ there is a unique spherical function $\omega_m \in \mathcal{S}(G, K)$ such that

$$[\gamma(f)](m) = \int_G f(g)\omega_m(g^{-1})d\mu_G(g),$$

$\forall f \in L^1(K\backslash G/K)$. γ is related to the spherical transform by $[\gamma(f)](m) = \hat{f}(\omega_m)$

Let us see m as a map

$$m : L^1(K\backslash G/K) \to \mathbb{C}$$

defined by $m(f) = [\gamma(f)](m)$. Then $|m(f)| \leq \|\gamma(f)\|_\infty \leq \|\psi(f)\| \leq \|f\|_1$ and
$m(f_1 * f_2) = [\gamma(f_1 * f_2)](m) = [\gamma(f_1)](m)[\gamma(f_2)](m) = m(f_1)m(f_2)$ and $m \neq 0$ because γ is one to one.

m is a multiplicative linear functional on $L^1(K\backslash G/K)$. In other words, we have $\omega_m \in \mathcal{S}(G, K)$ such that

$$m(f) = \int_G f(g)\omega_m(g^{-1})d\mu_G(g)$$

$\forall f \in L^2(K\backslash G/)$

Let Σ be the set of positive definite bounded spherical functions on the Gelfand pair (G, K). It can be topologised by any of four topologies listed in 2.3.4 to become a locally convex space. For Lie groups with

polynomial growth, all bounded spherical functions are positive definite, therefore any bounded spherical function for $SE(2)$ is positive definite. The next theorem is needed in the proof of theorem 3.6

3.5 Theorem ([10],theorem 8.2.7)
The continuous homomorphism $C_c(K\backslash G/K) \to \mathbb{C}$ (and $L^1(K\backslash G/K) \to \mathbb{C}$) are the maps $f \mapsto \int_G f(x)\varphi(x^{-1})d\mu_G(x)$ where φ is is a bounded (G,K) spherical function on G.

Next theorem is the main result of this paper

3.6 Theorem
Let φ be a spherical function for the pair (G,K) where $G = \mathbb{R}^n \rtimes SO(n)$ and $K = SO(n)$. Let $L^1(K\backslash G/K)$ be the Banach algebra of K-bi-invariant functions on G. Let Σ be the set of bounded positive spherical functions on G. For each $f \in L^1(K\backslash G/K)$, the spherical transform $\widehat{f}$ of f extends to a holomorphic function in Σ such that the map $f \to \hat{f}$ is an isomorphism of $L^1(K\backslash G/K)$ onto $\mathcal{PW}_\Sigma(G,K)$

Proof. Let $f : L^1(K\backslash G/K) \to \Sigma$, we are going to show that f is a homomorphism, linear, bijective and continuous. Thereafter, we prove that any $\varphi \in \Sigma$ extends holomorphically to functions on $\mathbb{C}$. To this end, the spherical transform of $\varphi \in L^1(K\backslash G/K)$ is defined as

$$\widehat{f}(\varphi) = \int_G f(g)\varphi(g^{-1})d\mu_G(g) = m_\varphi(f). \tag{4}$$

To establish the homomorphism, we are going to show that$(\widehat{f_1 * f_2})(\varphi) = \widehat{f}_1(\varphi)*\widehat{f}_2(\varphi)$ for $f_1, f_2 \in L^1(K\backslash G/K)$. now

$$\begin{aligned}
(\widehat{f_1 * f_2})(\varphi) &= \int_G (f_1 * f_2)(\varphi)\varphi(y^{-1})d\mu(y) \\
&= \int_G \int_G f_1(g) f_2(g^{-1}y)\varphi(y^{-1})d\mu(g)d\mu(y) \\
&= \int_G f_1(g)\left[\int_G f_2(g^{-1}y)\varphi(y^{-1})d\mu(y)\right]d\mu(g) \\
&Let z = g^{-1}y \Rightarrow gz = y \Rightarrow y^{-1} = z^{-1}g^{-1} \\
&= \int_G f_1(g)\left[\int_G f_2(z)\varphi(z^{-1}g^{-1})d\mu(z)\right]d\mu(g) \\
&change\ z = kz\ and\ integrate\ over\ k,\ but\ f(kz)\ =\ f(z) \\
&= \int_G f_1(g)\left[\int_G \int_K f_2(z)\varphi(z^{-1}k^{-1}g^{-1})d\mu(z)d\mu(k)\right]d\mu(g) \\
&since\ \int_G \varphi(g_1 k g_2)d\mu(g) = \varphi(g_1)\varphi(g_2),\ we\ have \\
&= \int_G f_1(g)\int_G f(z)d\mu(z)\left[\int_K \varphi(z^{-1}k^{-1}g^{-1})d\mu(k)\right]d\mu(g) \\
&= \int_G f_1(g)\int_G f(z)d\mu(z)\Big(\varphi(z^{-1})\varphi(g^{-1})d\mu(z)d\mu(g) \\
&= \int_G f_1(g)\varphi(g^{-1})d\mu(g)\int_G f(z)\varphi(z^{-1})d\mu(z) \\
&= \widehat{f}_1(\varphi)\widehat{f}_2(\varphi)
\end{aligned}$$

Theorem 3.5 establishes the continuity of f. Also, looking at definition 3.4, the Gelfand transform is injective therefore f is injective and since the unique image of $\varphi \in L^1(K\backslash G/K)$ lies in Σ it also means that f is surjective. Hence, f is an isomorphism. Let D^* denote the closed unit disk defined in prop. 3.2, it is closed and compact(see prop. 3.2) and $\mathcal{P}(G,K) \subset D^*$ (see prop. 3.2). Let us define $C_x = \{z \in \mathbb{C} \mid\mid z \mid\leq\mid\mid x \mid\mid\}$. Then $D^* \subset C$ where C has the product topology $C = \prod_{x\in\mathcal{A}} C_x$ (see prop. 4.6). Let h belong to the closure of D^* in C, then h is linear (from prop.3.2).
To prove that $\varphi \in \Sigma$ extends to a holomorphic function on $\mathbb{C}$, we need to show that φ is a series of entire function that converges uniformly on a compact set. Our spherical function for the Gelfand pair $(SO(2) \rtimes \mathbb{R}^2, SO(2))$ is the Bessel functin $J_0(\sqrt{\lambda r})$ of order zero. This function is known to be the cosine function(Cos$\sqrt{\lambda r}$). Our

spectrum, as mentioned earlier is the set of bounded spherical functions, which are analytic. Explicitly, they are Bessel functions of order zero $J_0(\lambda, r)$. Since e^z is an entire function, Cosine functions are entire functions. This implies that our spectrum is the set of entire functions on $\mathbb{C}$. It is locally compact and does not have group structure. It is topologized by the weak $*$ topology which is found to coincide with the Euclidean topology or the Compact open topology. With this topology on the spectrum, it becomes the space of entire functions on $\mathbb{C}$. Therefore, any function on Σ is also an entire function on $\mathbb{C}$. One important consequence of this result is that the space Σ is not larger than $L^1(K\backslash G/K)$. □

4. Conclusion

In this work, an explicit form of spherical function for $SE(2)$ is understood to be the Bessel function of order zero. The Gelfand spectrum for our result has been established to be the set of spherical functions on $SE(n)$ that are bounded and positive definite. This space is known to be isomorphic with $\mathbb{R}^+$. A topological isomorphism between the L^1 - algebra of K-bi-invariant functions on $SE(n)$ and the Gelfand spectrum has been established.

Conflict of Interest: The authors wish to state clearly that **there is no** conflict of interest in this research submitted for publication.
Compliance with Ethical Standards:
Funding: The Authors did not receive any fund for this research
Disclosure of potential conflicts of interest: The authors wish to state clearly that **there is no** conflict of interest in this research submitted for publication.
Data Availability Statements: This is not applicable because no data was used for this research.

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