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Phase Portraits in the Problem on Slipping of a Flexible Inextensible Chain with Dry Friction

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ABSTRACT
The plane problem on sliding down of an initially resting heavy flexible inextensible chain from a horizontal table with a rounded edge in the presence of dry friction is considered. An analytical formula for tension of the chain along its total length is found. Several phase portraits of the problem are depicted.

KEYWORDS: *flexible inextensible chain, chain tension, dry friction, phase portrait*

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I. INTRODUCTION

Let one part of a heavy, flexible and inextensible chain overhangs from a horizontal table having a rounded edge of the radius R. The other part lies on the table and the rectilinear piece of this part is perpendicular to the edge. The chain is entirely located in one vertical plane and it is assumed that all subsequent motions of the chain occur in this immovable plane. The distributed force of the Coulomb dry friction is applied throughout the entire contact of the chain with the support.

In substance, according to the papers [1], pp. 212-216, [2], [3] and others, it is often supposed that $R = 0$ i.e. a table with the sharp edge is considered in the problem. It was noted that, if the chain starts from the state of rest, the upper extremity of the chain has not enough time to reach the edge of the table because the contact constraint is unilateral and the chain leaves the support earlier. Moreover, this fact is true both in the frictionless and with friction cases. And the edge of the table may not be sharp, but rounded.

It should be noted, however, that in [2] the proof of these statements is incorrect. Namely, in Eq.(5), where the sum of the normal projections of the forces applied to an element of the chain is written, in the product of the normal pressure and R the latter cofactor is omitted by mistake.

The attempt to calculate the reaction on the table edge with $R \neq 0$ contains in [3]. But instead of eliminating the acceleration from the corresponding formula, the cofactor R has been equalled to zero.

Similar in formulation and results, the problem on motion of a heavy chain thrown over a smooth round pulley (Atwood machine) is examined in [5, 6].

A discrete analog of the problem on sliding down of a heavy inextensible chain from a smooth horizontal table is given in [7]. The chain is simulated by a set of material points having equal masses and connecting each with the neighboring points by massless rigid rods of an equal length. Numerical computations with different numbers of links specify the multi-link shape which is qualitatively very resembling to the experimental results published in the paper [8] where, for verification of theoretical models of the considered problem, video analysis tools are used.

More general and advanced models of a flexible chain accepted as one-dimensional continuous medium are proposed and examined in [9, 10]. Herewith, the impacts, arising in the system with the sharp edge $R = 0$ due to a quick change of the velocity vector direction of the chain element at the point of its contact with the table sharp edge, are taken into account. In [9], the penetration and propagation of impacts in the medium are described using the equation of the balance of the so-called material quantity of the chain motion. In [10], for such a discription the generalization of the Rayleigh principle of energy dissipation is given and the theory of dissipation singularities in a one-dimensional inextensible continuum (string) is constructed.

Below, in the frames of the traditional classic model of a flexible inextensible chain the problem with the rounded table edge in the presence of dry friction is treated.

II. FORMULA FOR THE CHAIN TENSION

Suppose that one extremity of the chain hangs down freely from the rounded table edge. At three parts of the chain (Fig. 1) its motion is described by the different equations. The abscissa of the upper chain extremity A is $x < 0$. The length s of the chain is counted from this extremity.

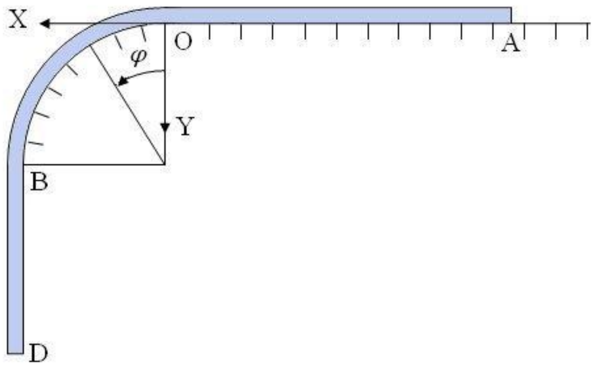

Figure 1: Chain slips off a horizontal tablehaving the rounded edge

The equation of an inseparable motion of the chain at the part AO takes the form

$$-\ \rho x\ddot{x} = k\rho xg + T_O$$

where ρ is the unit length mass of the chain, k is the coefficient of dry friction, g is the acceleration due to gravity of a free falling body at a place in vacuum, T_O is the force of tension applied to the segment AO of the chain at the point O. Denoting by v the speed $\dot{x}$ rewrite this equation in the form

$$x\dot{v} = -kxg - \rho^{-1}T_O \quad (1)$$

At the part OB the equations of an inseparable motion of the chain take the form ([11], p. 303)

$$\rho\dot{v} = \rho g\sin\varphi + \frac{\partial T}{\partial s} - kN,\ \ \rho\frac{v^2}{R} = \rho g\cos\varphi + \frac{T}{R} - N \quad (2)$$

$\varphi = (s + x)/R$ is the angle between OY-axis and the principal normal of the circle arc OB, $T(s,\varphi)$ is the chain tension at the point with the Lagrange coordinate $s\ (-x \le s \le -x + a)$, N is the force of the support normal pressure refered to the unit length and applied to the chain at the given point.

And finally, at the vertical part BD the chain motion is described by the equation

$$l\dot{v} = gl - \rho^{-1}T_B \quad (3)$$

where T_B is the chain tension force applied to the segment BD at the point B.

The following notations are introduced here($L$is the total chain length)

$$l = L + x - a > 0,\quad a = \pi R/2$$

From Eqs.(2) we find

$$\frac{\partial T}{\partial s} - \frac{k}{R}T = W,\qquad W = \rho\left[\dot{v} - \frac{kv^2}{R} - g(\sin\varphi - k\cos\varphi)\right]$$

where $W = W(s,t)$. Hence

$$\underset{-x\,\le s\,\le -x+a}{T(s,t)} = \frac{\rho R}{k}\left\{\frac{kv^2}{R} - \dot{v} + \frac{g}{1+k^2}\left[k(1-k^2)\cos\varphi + 2k^2\sin\varphi\right]\right\} + C(t)e^{k\varphi} \quad (4)$$

The functions $\dot{v}, C(t)$ of the time we find from the Eqs. (1) and (4).

$$\dot{v} = k\left[g\left(L + (1+jk)x - \frac{\pi R}{2} + R\frac{j(1-k^2)-2k}{1+k^2}\right) + (j-1)v^2\right]Z^{-1} \quad (5)$$

$$C(t) = \rho Q + \rho(kx - R)(M + jQ)Z^{-1}$$

After excluding the specified functions in the formula (5), as the result, we obtain the following formula for the chain tension at the point with the internal coordinate s on the rounded table edge as the function of $x(t), v(t)$ and the parameters ρ, R, k, l of the system

$$\underset{-x\le s\le -x+a}{T(s,t)} = \rho e^{k\varphi}\left[Q + (kx - R)(M + jQ)Z^{-1}\right] + \quad (6)$$

$$+\ \rho\{v^2 + R(M + jQ)Z^{-1} + Rg[(1-k^2)\cos\varphi + 2k\sin\varphi](1+k^2)^{-1}\}$$

Here we denote

$$j = \exp\left(\frac{k\pi}{2}\right),\ \ Q = \frac{g\left[(k^2-1)R - kx(1+k^2)\right]}{1+k^2} - v^2,\ \ M = \frac{g(2kR - (1+k^2)\,l}{1+k^2} + v^2,\ \ Z = k(l - jx) + R(j-1)$$

With $x < 0$ the denominator Z is nonzero ($Z > 0$).

III. NECESSARY CONDITIONS FOR THE ONSET OF CHAIN MOTION FROM THE STATE OF REST

Due to gravity the rolling down force $\rho gR\sin\varphi d\varphi$ and the pressure $\rho gR\cos\varphi d\varphi$ act upon a chain element ρds that supports on the edge arc OB small part. Hence, the total pulling down force equals $\rho g\ (l +$

R)and the total force of resistance is $k\rho g\,(R-x)+k\int_0^{\pi/2} Td\varphi$. So for the onset of chain motion from the state of rest it is necessary that the condition

$$l > R(k-1) - kx + \frac{k}{\rho g}\int_0^{\frac{\pi}{2}} Td\varphi \qquad (7)$$

be valid where T is given by (6) with $v = 0$.

Of course, it is also necessary that $\dot{v} > 0$ at the start instant. From the Eq. (5) it follows that, if all the values are fixed except of k, the magnitude of the right-hand side decreases when the coefficient k increases. In this case, the sign of the right-hand side becomes negative if the initial value $-x > 0$ is great enough.

The Eqs. (1-3) describe chain motion correctly for all three parts of the chain until the point A arrives to the position O or contact at some point of the arc OB disappears.

From Eq. $(2)_2$ and the formula (6) we find the expression for the force N which acts upon the chain at the points of the arc OB and is referred to the unit of the chain length

$$N = \rho\Big\{e^{k\varphi}\Big[g\Big(l(1+k)(R-kx) + Rx(2+k) - R^2 + 2R\frac{R-x-k(R+l)}{1+k^2}\Big) +$$
$$+\ kv^2(x-l)\Big] +\ R[g(j(R-kx)-l) + v^2(1-j)] + 2gR^2\frac{k-j}{1+k^2}\Big\}(RZ)^{-1} +$$
$$+\ \frac{2\rho g}{\sqrt{1+k^2}}\cos(\varphi-\alpha) \qquad (0 \le \varphi \le \pi/2,\quad \tan\alpha = k)$$

Inside the interval $(0,\pi/2)$ the function $N(\varphi)$ can have only one extremum since, if there exist an inner critical point $\varphi = \varphi_{extr}$ of this function, then the condition

$$\frac{d^2N}{d\varphi^2} = -2\rho g\cos\varphi_{extr} < 0$$

fulfills at this point and this extremum is maximum. Hence, in any case, the continuous non-constant function $N(\varphi)$ can posess the single minimal value at the extremities O or/and B.

So the inequality between values of N_O and N_B specifies weakening of the chain contact firstly at the point O ($N_O = 0$) or at the point B ($N_B = 0$).

In the frictionless case the contact at the point B always weakens earlier than at the point O [12].

In the presence of friction both the inequalities $N_O < N_B$ and $N_O > N_B$ can be true. It depends on values of the parameters including the coefficient k of friction. Below, in the numerical example, the difference of pressure at points O and B as function of x is depicted in Fig. 6 for the case of $k = 0.5$ and $v = 0$.

The instant, when, for the first time, the contact at one of the extremities of the segment OB disappears, is used for stopping the numerical computations in the next section.

IV. NUMERICAL EXAMPLE

Consider the numerical example

$$\rho = 0.01\ kg/m\,,\quad L = 12\ m\,,\quad x_0 \ge -8\ m\,,\quad R = 2\ m\,,\quad g = 9.8\ m/s^2$$

The phase portrait for frictionless case is depicted in Fig. 2. With the energy integral the graph is plotted simply. At the upper endpoints the condition $N_B = 0$ is fulfilled and then the chain leaves the support. Note that the point A does not reach the table edge except of the cases when in the start position of the extremity A lies sufficiently close to the point O.

In Fig. 3 the phase portrait is given for sufficiently small value of the friction coefficient ($k = 0.2$). It is noticeable that, in the presence of friction, the extremity A approaches closer to the position O. Moreover, as before, the condition $N_B = 0$ happens at the upper endpoints segments of graphs if $x < 0$ at these points.

Let the friction coefficient be $k = 0.5$. Fig. 4 depicts the graphs of the functions (with the condition $v = 0$ fulfilled)

$$f(x) = (1+k)x + R(1-k) + L - a - \frac{k}{\rho g}\int_0^{\pi/2} Td\varphi$$

(the thin line) and $\dot{v}(x)$ plotted according to the formula (5). The curves intersect the abscissae axis at the point $x_0 \approx -4.72$. Hence for any starting point $x > x_0$ the resulting force pulls the chain down over the edge of the table and the condition (7) is valid.

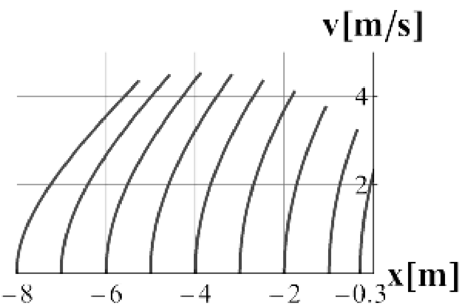

Figure 2: $k = 0$

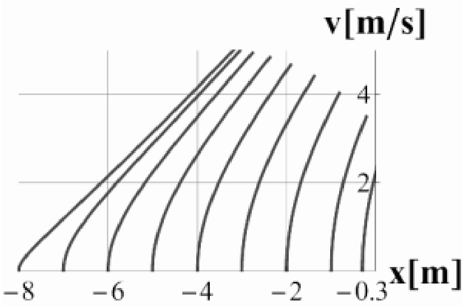

Figure 3: $k = 0.2$

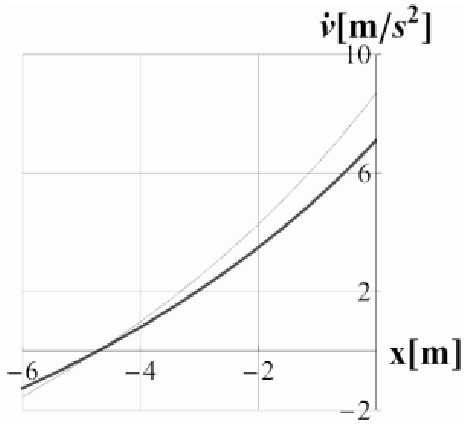

Figure 4: Graphs of the necessary condition for the onset of motion

The subsequent motion of the chain occurs similarly to the case of low friction, see Fig. 5. In comparison with Fig. 3, it is noticeable that, while the friction coefficient increases, the extremity A of the chain moves even further in the direction to the table edge O and gains a greater speed during the contact motion up to the instant when the contact at the point B weakens ($N_B = 0$).

By the way, in contrast to the frictionless case, when the friction coefficient is not too small the inequality $N_O > N_B$ can not be valid. In Fig. 6, the graph of the difference in pressure values at points O and B is plotted under the conditions $k = 0.5$ and $v = 0$. At the instant of the beginning of motion, when $0 > x > -4.29$, the inequality $N_O < N_B$ is true. By continuity, it is also valid in some interval for small values of $v > 0$.

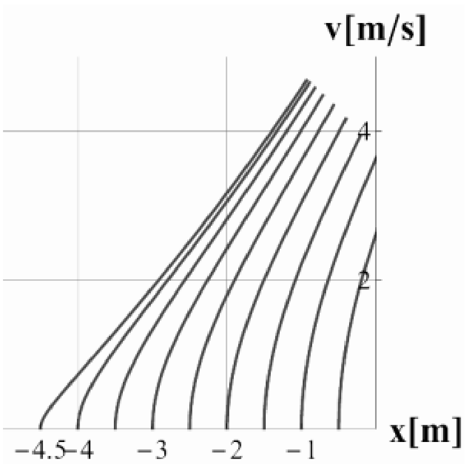

Figure 5: $k = 0.5$

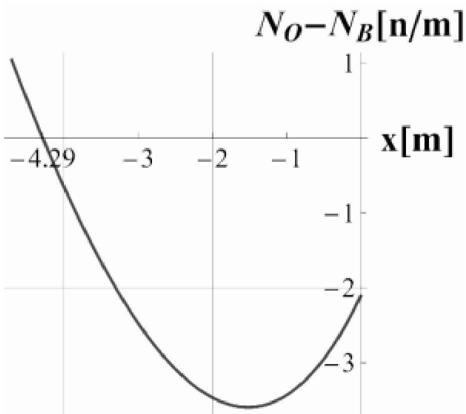

Figure 6: $k = 0.5$,$N_O < N_B$ with $v = 0$

V. CONCLUSION

The simple mechanical problem considered in this paper demonstrates once more that in the presence of dry friction motions of systems possess more rich properties and pecularities against the frictionless case.

REFERENCE

[1]. E. Lainé, Exercices de Mécanique, Librarie Vuibert, Paris. 1964.

[2]. Juan R.Sanmartin, Miguel. A.Vallejo, Widespread error in a standard problem in the dynamics of deformable bodies, Am. J. Phys., 1978, vol. 46, no. 9, 949-950.

[3]. D. Prato, R.J. Gleiser R.J., Another look at the uniform rope sliding over the edge of a smooth table, Am. J. Phys., 1982, vol. 50, no. 6,536-539.

[4]. J.R.Sanmartin, M.A. Vallejo, Comment on "Another look at the uniform rope sliding over the edge of a smooth table", Am. J. Phys., 1983, vol. 51, no. 7, p. 585.

[5]. M.G. Calcin, The dynamics of a falling chain: II, Am. J. Phys., 1989, vol. 57, no. 2, 157-159.

[6]. P.T.Brun, B. Audoly, A. Goriely &D. Vella, The surprising dynamics of a chain on a pulley: lift off and snapping // Proc. R. Soc.London. Ser. A. **472**: 20160187 (2016). http://dx.doi.org/10.1098/rspa.2016.0187

[7]. Jan Vrbik, Chain sliding off a table, Am. J. Phys., 1993, vol. 61, no. 3, 258-261.

[8]. R. Moreno, A.Page, J.Riera &J.L. Hueso, Video analysis of sliding chains: A dynamic model based on variable-mass systems, Am. J. Phys., 2015, vol. 83, no. 6, 258-261.

[9]. O.M. O'Reilly, Modeling Nonlinear Problems in the Mechanics of Strings and Rods (The Role of the Balance Laws),Springer, 2017, XX+425 p.

[10]. E.G. Virga, Chain paradoxes, Proc. R. Soc. London. Ser. A. **471**: 20140657 (2014). http://dx.doi.org/10.1098/rspa.2014.0657

[11]. A.E.H.Love, Theoretical Mechanics (An Introductory Treatise on the Principles of Dynamics), Cambridge: Univ. Press, 1897, XVI+379 p.

[12]. A.S. Sumbatov, Plane-Parallel Sliding of a Flexible Inextensible Chain over the Rounded Edge of a Horizontal Table, Mech. Solids, 2021, vol. 56, no. 8, 1569-1577.

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