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Combinatorial Identities Related To Root Supermultiplicities In Some Borcherds Superalgebras

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ABSTRACT
In this paper, root supermultiplicities and corresponding combinatorial identities for the Borcherds superalgebras which are extensions of B_2 and B_3 are found out.

Keywords: Borcherds superalgebras, Colored superalgebras, Root supermultiplicities.

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I. INTRODUCTION

The theory of Lie superalgebras was constructed by Kac in 1977. The theory of Lie superalgebras can also be seen in Scheunert(1979) in a detailed manner. The notion of Kac-Moody superalgebras was introduced by Kac(1978) and therein the Weyl-Kac character formula for the irreducible highest weight modules with dominant integral highest weight which yields a denominator identity when applied to 1-dimensional representation was also derived. Borcherds(1988, 1992) proved a character formula called Weyl-Borcherds formula which yields a denominator identity for a generalized Kac-Moody algebra.Kang developed homological theory for the graded Lie algebras in 1993a and derived a closed form root multiplicity formula for all symmetrizable generalized Kac-Moody algebras in 1994a. Miyamoto(1996) introduced the theory of generalized Lie superalgebra version of the generalized Kac-Moody algebras(Borcherds algebras). Kim and Shin(1999) derived a recursive dimension formula for all graded Lie algebras. Kang and Kim(1999) computed the dimension formula for graded Lie algebras. Computation of root multiplicities of many Kac-Moody algebras and generalized Kac-Moody algebras can be seen in Frenkel and Kac(1980), Feingold and Frenkel(1983),Kass et al.(1990), Kang(1993b, 1994b,c,1996), Kac and Wakimoto(1994), Sthanumoorthy and Uma Maheswari(1996b), Hontz and Misra(2002), Sthanumoorthy et al.(2004a,b) and Sthanumoorthy and Lilly(2007b). Computation of root multiplicities of Borcherds superalgebras was found in Sthanumoorthy et al.(2009a). Some properties of different classes of root systems and their classifications for Kac-Moody algebras and Borcherds Kac-Moody algebras were studied in Sthanumoorthy and Uma Maheswari(1996a) and Sthanumoothy and Lilly(2000, 2002,2003,2004, 2007a). Also, properties of different root systems and complete classifications of special, strictly, purely imaginary roots of Borcherds Kac-Moody Lie superalgebras which are extensions of Kac-Moody Lie algebras were explained in Sthanumoorthy et al.(2007,2009b) and Sthanumoorthy and Priyadharsini(2012, 2013). Moreover, Kang(1998) obtained a superdimension formula for the homogeneous subspaces of the graded Lie superalgebras, which enabled one to study the structure of the graded Lie superalgebras in a unified way. Using the Weyl-Kac-Borcherds formula and the denominator identity for the Borcherds superalgerbas, Kang and Kim(1997) derived a dimension formula and combinatorial identities for the Borcherds superalgebras and found out the root multiplicities for Monstrous Lie superalgebras. Borcherds superalgebras which are extensions of Kac Moody algebras A_2 and A_3 were considered in Sthanumoorthy et al.(2009a) and therein dimension formulas were found out.In Sthanumoorthy and Priyadharsini(2014), we have computed combinatorial identities for A_2 and A_3 and root super multiplicities for some hyperbolic Borcherds superalgebras.

The aim of this paper is to compute dimensional formulae, root supermultiplicities and corresponding combinatorial identities for the Borcherds superalgebras which are extensions of Kac-Moody algebras B_2 and B_3.

II. PRELIMINARIES

In this section, we give some basic concepts of Borcherds superalgebras as in Kang and Kim (1997).

Definition 2.1: Let I be a countable (possibly infinite) index set. A real square matrix $A = (a_{ij})_{i,j\in I}$ is called Borcherds-Cartan matrix if it satisfies:

(1) $a_{ii} = 2$ or $a_{ii} \le 0$ for all $i \in I$,

(2) $a_{ij} \le 0$ if $i \ne j$ and $a_{ij} \in \mathbb{Z}$ if $a_{ii} = 2$,

(3) $a_{ij} = 0 \Leftrightarrow a_{ji} = 0$

We say that an index i is real if $a_{ii} = 2$ and imaginary if $a_{ii} \le 0$. We denote by

$$I^{re} = \{i \in I \mid a_{ii} = 2\},\quad I^{im} = \{i \in I \mid a_{ii} \le 0\}.$$

Let $\underline{m} = \{m_i \in \mathbb{Z}_{>0} \mid i \in I\}$ be a collection of positive integer such that $m_i = 1$ for all $i \in I^{re}$. We call $\underline{m}$ a charge of A.

Definition 2.2: A Borcherds-Cartan matrix A is said to be symmetrizable if there exists a diagonal matrix $D = \mathrm{d}iag\,(\varepsilon_i; i \in I)$ with $\varepsilon_i > 0\ (i \in I)$ such that DA is symmetric.

Let $C = (c_{ij})_{i,j\in I}$ be a complex matrix satisfying $c_{ij}c_{ji} = 1$ for all $i, j \in I$. Therefore, we have $c_{ii} = \pm 1$ for all $i \in I$. We call $i \in I$ an even index if $c_{ii} = 1$ and an odd index if $c_{ii} = -1$.

We denote by I^{even} (I^{odd}) the set of all even (odd) indices.

Definition 2.3: A Borcherds-cartan matrix $A = (a_{ij})_{i,j\in I}$ is restricted (or colored) with respect to C if it satisfies:

If $a_{ii} = 2$ and $c_{ii} = -1$ then a_{ij} are even integers for all $j \in I$. In this case, the matrix C is called a coloring matrix of A.

Let $\mathfrak{h} = (\oplus_{i\in I}\mathbb{C}h_i) \oplus (\oplus_{i\in I}\mathbb{C}d_i)$ be a complex vector space with a basis $\{h_i, d_i; i \in I\}$, and for each $i \in I$, define a linear functional $\alpha_i \in \mathfrak{h}^*$ by

$$\alpha_i(h_j) = a_{ji},\ \alpha_i(d_j) = \delta_{ij}\quad for\ \ all\quad j \in I \text{..........\quad......}(2.1)$$

If A is symmetrizable, then there exists a symmetric bilinear form $(|)$ on $\mathfrak{h}^*$ satisfying $(\alpha_i \mid \alpha_j) = \varepsilon_i a_{ij} = \varepsilon_j a_{ji}$ for all $i, j \in I$.

Definition 2.4: Let $Q = \oplus_{i\in I}\mathbb{Z}\alpha_i$ and $Q_+ = \sum_{i\in I}\mathbb{Z}_{\ge 0}\alpha_i$, $Q_- = -Q_+$. Q is called the root lattice.

The root lattice Q becomes a (partially) ordered set by putting $\lambda \ge \mu$ if and only if $\lambda - \mu \in Q_+$.

The coloring matrix $C = (c_{ij})_{i,j\in I}$ defines a bimultiplicative form $\theta : Q \times Q \to C^X$ by

$$\theta(\alpha_i, \alpha_j) = c_{ij}\quad for\ \ all\,i, j \in I,$$

$$\theta(\alpha + \beta, \gamma) = \theta(\alpha, \gamma)\theta(\beta, \gamma),$$

$$\theta(\alpha, \beta + \gamma) = \theta(\alpha, \beta)\theta(\alpha, \gamma)$$

for all $\alpha, \beta, \gamma \in Q$. Note that θ satisfies

$$\theta(\alpha, \beta)\theta(\beta, \alpha) = 1\ \ forall\ \ \alpha, \beta \in Q, \text{.......... ....}(2.2)$$

since $c_{ij}c_{ji} = 1$ for all $i, j \in I$. In particular $\theta(\alpha, \alpha) = \pm 1$ for all $\alpha \in Q$.

We say $\alpha \in Q$ is even if $\theta(\alpha, \alpha) = 1$ and odd if $\theta(\alpha, \alpha) = -1$.

Definition 2.5: A θ-colored Lie super algebra is a Q-graded vector space $L = \oplus_{\alpha\in Q} L_\alpha$ together with a bilinear product $[,] : L \times L \to L$ satisfying

$$[L_\alpha, L_\beta] \subset L_{\alpha+\beta},$$

$$[x, y] = -\theta(\alpha, \beta)[y, x],$$

$$[x,[y,z]] = [[x,y],z] + \theta(\alpha,\beta)[y,[x,z]]$$

for all $\alpha,\beta \in Q$ and $x \in L_\alpha$, $y \in L_\beta$, $z \in L$.

In a θ-colored Lie super algebra $L = \oplus_{\alpha\in Q} L_\alpha$, for $x \in L_\alpha$, we have $[x,x]=0$ if α is even and $[x,[x,x]] = 0$ if α is odd.

Definition 2.6: The universal enveloping algebra $U(L)$ of a θ-colored Lie super algebra L is defined to be $T(L)/J$, where $T(L)$ is the tensor algebra of L and J is the ideal of $T(L)$ generated by the elements $[x,y] - x\otimes y + \theta(\alpha,\beta) y\otimes x$ $(x \in L_\alpha, y \in L_\beta)$.

Definition 2.7: The Borcherds superalgebra $\mathfrak{g} = \mathfrak{g}(A,\underline{m},C)$ associated with the symmetrizable Borcherds-Cartan matrix A of charge $\underline{m} = (m_i; i \in I)$ and the coloring matrix $C = (c_{ij})_{i,j\in I}$ is the θ-colored Lie super algebra generated by the elements $h_i, d_i (i \in I)$, $e_{ik}, f_{ik} (i \in I, k = 1,2, \cdots, m_i)$ with defining relations:

$$\begin{aligned}
[h_i, h_j] &= [h_i, d_j] = [d_i, d_j] = 0,\\
[h_i, e_{jl}] &= a_{ij} e_{jl}, [h_i, f_{jl}] = -a_{ij} f_{jl},\\
[d_i, e_{jl}] &= \delta_{ij} e_{jl}, [d_i, f_{jl}] = -\delta_{ij} f_{jl},\\
[e_{ik}, f_{jl}] &= \delta_{ij}\delta_{kl} h_i\\
(ade_{ik})^{1-a_{ij}} e_{jl} &= (adf_{ik})^{1-a_{ij}} f_{jl} = 0 \ if \ a_{ii} = 2 \ and \ i \neq j,\\
[e_{ik}, e_{jl}] &= [f_{ik}, f_{jl}] = 0 \ if \ a_{ij} = 0
\end{aligned}$$

for $i, j \in I, k = 1, \cdots, m_i, l = 1, \cdots, m_j$.

The abelian subalgebra $\mathfrak{h} = (\oplus_{i\in I} \mathbb{C} h_i) \oplus (\oplus_{i\in I} \mathbb{C} d_i)$ is called the Cartan subalgebra of $\mathfrak{g}$ and the linear functionals $\alpha_i \in \mathfrak{h}^* (i \in I)$ defined by (2.1) are called the simple roots of $\mathfrak{g}$. For each $i \in I^{re}$, let $r_i \in GL(\mathfrak{h}^*)$ be the simple reflection of $\mathfrak{h}^*$ defined by

$$r_i(\lambda) = \lambda - \lambda(h_i)\alpha_i \quad (\lambda \in \mathfrak{h}^*).$$

The subgroup W of $GL(\mathfrak{h}^*)$ generated by the r_i's $(i \in I^{re})$ is called the Weyl group of the Borcherds super algebra $\mathfrak{g}$.

The Borcherds superalgebra $\mathfrak{g} = \mathfrak{g}(A,\underline{m},C)$ has the root space decomposition $\mathfrak{g} = \oplus_{\alpha\in Q} \mathfrak{g}_\alpha$, where

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h,x] = \alpha(h)x \ for \ all \ h \in \mathfrak{h}\}.$$

Note that $\mathfrak{g}_{\alpha_i} = \mathbb{C} e_{i,1} \oplus \cdots \oplus \mathbb{C} e_{i,m_i}$

and $\mathfrak{g}_{-\alpha_i} = \mathbb{C} f_{i,1} \oplus \cdots \oplus \mathbb{C} f_{i,m_i}$

We say that $\alpha \in Q^X$ is a root of $\mathfrak{g}$ if $\mathfrak{g}_\alpha \neq 0$. The subspace $\mathfrak{g}_\alpha$ is called the root space of $\mathfrak{g}$ attached to α. A root α is called real if $(\alpha \mid \alpha) > 0$ and imaginary if $(\alpha \mid \alpha) \leq 0$.

In particular, a simple root α_i is real if $a_{ii} = 2$ that is if $i \in I^{re}$ and imaginary if $a_{ii} \leq 0$ that is if $i \in I^{im}$. Note that the imaginary simple roots may have multiplicity > 1. A root $\alpha > 0$ $(\alpha < 0)$ is called positive (negative). One can show that all the roots are either positive or negative. We denote by Δ, Δ_+ and Δ_- the set of all roots, positive roots and negative roots, respectively. Also we denote $\Delta_{\bar{0}}$ $(\Delta_{\bar{1}})$ the set of all even (odd) roots of $\mathfrak{g}$. Define the subspaces $\mathfrak{g}^{\pm} = \oplus_{\alpha\in\Delta_\pm} \mathfrak{g}_\alpha$.

Then we have the triangular decomposition of $\mathfrak{g}$: $\mathfrak{g} = \mathfrak{g}^- \oplus \mathfrak{h} \oplus \mathfrak{g}^+$.

Definition 2.8:(Sthanumoorthy et al.(2009b)) We define an indefinite nonhyperbolic Borcherds - Cartan matrix A, to be of extended-hyperbolic type, if every principal submatrix of A is of finite, affine, or hyperbolic type Borcherds - Cartan matrix. We say that the Borcherds superalgebra associated with a Borcherds - Cartan matrix A is of extended-hyperbolic type, if A is of extended-hyperbolic type.

Definition 2.9: A $\mathfrak{g}$-module V is called $\mathfrak{h}$-*diagonalizable*, if it admits a weight space decomposition $V = \bigoplus_{\mu \in \mathfrak{h}^*} V_\mu$, where $V_\mu = \{v \in V \mid h \cdot v = \mu(h)v \text{ for all } h \in \mathfrak{h}\}$.

If $V_\mu \neq 0$, then μ is called a *weight* of V, and $\dim V_\mu$ is called the multiplicity of μ in V.

Definition 2.10: A $\mathfrak{h}$-diagonalizable $\mathfrak{g}$-module V is called a *highest weight module* with highest weight $\lambda \in \mathfrak{h}^*$, if there is a nonzero vector $v_\lambda \in V$ such that

1. $e_{ik} \cdot v_\lambda = 0$, for all $i \in I$, $k = 1, \cdots, m_i$,
2. $h \cdot v_\lambda = \lambda(h) v_\lambda$ for all $h \in \mathfrak{h}$ and
3. $V = U(\mathfrak{g}) \cdot v_\lambda$.

The vector v_λ is called a *highest weight vector*.

For a highest weight module V with highest weight λ, we have

(i) $V = U(\mathfrak{g}^-) \cdot v_\lambda$,

(ii) $V = \oplus_{\mu \le \lambda} V_\mu$, $V_\lambda = \mathbb{C} v_\lambda$ and

(iii) $\dim V_\mu < \infty$ for all $\mu \le \lambda$.

Definition 2.11: Let $P(V)$ denote the set of all weights of V. When all the weights spaces are finite dimensional, the character of V is defined to be $\mathrm{ch} V = \sum_{\mu \in \mathfrak{h}^*} (\dim V_\mu) e^\mu$,

where e^μ are the basis elements of the group $\mathbb{C}[\mathfrak{h}^*]$ with the multiplication given by $e^\mu e^\nu = e^{\mu+\nu}$ for $\mu, \nu \in \mathfrak{h}^*$. Let $b_+ = \mathfrak{h} \oplus \mathfrak{g}_+$ be the Borel subalgebra of $\mathfrak{g}$ and $\mathbb{C}_\lambda$ be the 1-dimensional b_+-module defined by $\mathfrak{g}_+ \cdot 1 = 0$, $h \cdot 1 = \lambda(h)1$ for all $h \in \mathfrak{h}$. The induced module $M(\lambda) = U(\mathfrak{g}) \otimes_{U(b_+)} \mathbb{C}_\lambda$ is called the *Verma module* over $\mathfrak{g}$ with highest weight λ. Every highest weight $\mathfrak{g}$-module with highest weight λ is a homomorphic image of $M(\lambda)$ and the Verma module $M(\lambda)$ contains a unique maximal submodule $J(\lambda)$. Hence the quotient $V(\lambda) = M(\lambda)/J(\lambda)$ is irreducible.

Let P^+ be the set of all linear functionals $\lambda \in \mathfrak{h}^*$ satisfying

$$\begin{cases} \lambda(h_i) \in \mathbb{Z}_{\ge 0} & \text{for all } i \in I^{re} \\ \lambda(h_i) \in 2\mathbb{Z}_{\ge 0} & \text{for all } i \in I^{re} \cap I^{odd} \\ \lambda(h_i) \ge 0 & \text{for all } i \in I^{im} \end{cases}$$

The elements of P^+ are called the dominant integral weights.

Let $\rho \in \mathfrak{h}^*$ be the $\mathbb{C}$-linear functional satisfying $\rho(h_i) = \frac{1}{2} a_{ii}$ for all $i \in I$. Let T denote the set of all imaginary simple roots counted with multiplicities, and for $F \subset T$, we write $F \perp \lambda$, if $\lambda(h_i) = 0$ for all $\alpha_i \in F$.

Definition 2.12:[Kang and Kim (1997)] Let J be a finite subset of I^{re}. We denote by

$\Delta_J = \Delta \cap (\sum_{j\in J} \mathbb{Z}\alpha_j)$, $\Delta_J^{\pm} = \Delta^{\pm} \cap \Delta_J$ and $\Delta^{\pm}(J) = \Delta^{\pm} \setminus \Delta_J^{\pm}$. Let

$$\mathfrak{g}_0^{(J)} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha\in\Delta_J} \mathfrak{g}_\alpha \right), \quad (2.3)$$

and

$$\mathfrak{g}_{\pm}^{(J)} = \bigoplus_{\alpha\in\Delta^{\pm}(J)} \mathfrak{g}_\alpha .$$

Then $\mathfrak{g}_0^{(J)}$ is the restricted Kac-Moody super algebra (with an extended Cartan subalgebra) associated with the Cartan matrix $A_J = (a_{ij})_{i,j\in J}$ and the set of odd indices $J^{odd} = J \cap I^{odd}$

$= \{ j \in J \mid c_{jj} = -1\}$

Then the *triangular decomposition* of $\mathfrak{g}$ is given by $\mathfrak{g} = \mathfrak{g}_-^{(J)} \oplus \mathfrak{g}_0^{(J)} \oplus \mathfrak{g}_+^{(J)}$.

Let $W_J = \langle r_j \mid j \in J \rangle$ be the subgroup of W generated by the simple reflections r_j $(j \in J)$, and let

$$W(J) = \{w \in W \mid \Delta_w \subset \Delta^+(J)\}$$

where $\Delta_w = \{\alpha \in \Delta^+ \mid w^{-1}\alpha < 0\}$.(2.4)

Therefore W_J is the Weyl group of the restricted Kac-Moody super algebra $\mathfrak{g}_0^{(J)}$ and $W(J)$ is the set of right coset representatives of W_J in W. That is $W = W_J W(J)$.

The following lemma given in Kang and Kim (1999), proved in Liu (1992), is very useful in actual computation of the elements of W(J).

Lemma 2.13: Suppose $w = w' r_j$ and $l(w) = l(w') + 1$. Then $w \in W(J)$ if and only if $w' \in W(J)$ and $w'(\alpha_J) \in \Delta^+(J)$.

Let $\Delta_{\bar{i},J}^{\pm} = \Delta_{\bar{i}}^{\pm} \cap \Delta_J\ (i = 0,1)$ and $\Delta_{\bar{i}}^{\pm}(J) = \Delta_{\bar{i}}^{\pm} \setminus \Delta_{\bar{i},J}^{\pm}\ (i = 0,1)$. Here $\Delta_{\bar{0}}^{\pm}(\Delta_{\bar{1}}^{\pm})$ denotes the set of all positive or negative even (resp., positive or negative odd) roots of g.

The following proposition, proved in Kang and Kim (1997), gives the denominator identity for Borcherds superalgebras.

Proposition 2.14: [Kang and Kim (1997)]. Let J be a finite subset of the set of all real indices I^{re}. Then

$$\frac{\prod_{\alpha\in\Delta_{\bar{0}(J)}^{-}} (1 - e^{\alpha})^{\dim \mathfrak{g}_\alpha}}{\prod_{\alpha\in\Delta_{\bar{1}(J)}^{-}} (1 + e^{\alpha})^{\dim \mathfrak{g}_\alpha}} = \sum_{\substack{w\in W(J) \\ F\subset T}} (-1)^{l(w)+|F|}\, \mathrm{ch} V_J(w(\rho - s(F)) - \rho))$$

where $V_J(\mu)$ denotes the irreducible highest weight module over the restricted Kac-Moody super algebra $\mathfrak{g}_0^{(J)}$ with highest weight μ and where F runs over all the finite subsets of T such that any two elements of F are mutually perpendicular. Here $l(w)$ denotes the length of w, $|F|$ the number of elements in F, and $s(F)$ the sum of the elements in F.

Definition 2.15: A basis elements of the group algebra $\mathbb{C}[\mathfrak{h}^*]$ by defining $E^{\alpha} = \theta(\alpha,\alpha)e^{\alpha}$.

Also define the super dimension $Dim\ \mathfrak{g}_\alpha$ of the root space $\mathfrak{g}_\alpha$ by $Dim\ \mathfrak{g}_\alpha = \theta(\alpha,\alpha) \dim \mathfrak{g}_\alpha$ .... (2.5)

Since $w(\rho - s(F)) - \rho$ is an element of Q_-, all the weights of the irreducible highest weight $\mathfrak{g}_0^{(J)}$-module $V_J(w(\rho - s(F)) - \rho)$ are also elements of Q_-.

Hence one can define the super dimension $DimV_{\mu}$ of the weight space V_{μ} of $V_J(w(\rho - s(F)) - \rho)$ in a similar way. More generally, for an $\mathfrak{h}$-diagonalizable $\mathfrak{g}_0^{(J)}$-module $V = \oplus_{\mu \in \mathfrak{h}^*} V_{\mu}$ such that $P(V) \subset Q$, we define the super dimension $DimV_{\mu}$ of the weight space V_{μ} to be

$$DimV_{\mu} = \theta(\mu, \mu) \dim V_{\mu} \quad (2.6)$$

For each $k \geq 1$, let

$$H_k^{(J)} = \bigoplus_{\substack{w \in W(J) \\ F \subset T \\ l(w)+|F|=k}} V_J(w(\rho - s(F)) - \rho) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.7)$$

and define the homology space $H^{(J)}$ of $\mathfrak{g}_-^{(J)}$ to be

$$H^{(J)} = \sum_{k=1}^{\infty} (-1)^{k+1} H_k^{(J)} = H_1^{(J)} \Theta H_2^{(J)} \oplus H_3^{(J)} \Theta \cdots, \quad (2.8)$$

an alternating direct sum of the vector spaces.

For $\tau \in Q_-$, define the super dimension $DimH_{\tau}^{(J)}$ of the τ-weight space of $H^{(J)}$ to be

$$DimH_{\tau}^{(J)} = \sum_{k=1}^{\infty} (-1)^{k+1} (DimH_k^{(J)})_{\tau}$$

$$= \sum_{k=1}^{\infty} (-1)^{k+1} \sum_{\substack{w \in W(J) \\ F \subset T \\ l(w)+|F|=k}} DimV_J(w(\rho - s(F)) - \rho)_{\tau}$$

$$= \sum_{\substack{w \in W(J) \\ F \subset T \\ l(w)+|F| \geq 1}} (-1)^{l(w)+|F|+1} DimV_J(w(\rho - s(F)) - \rho)_{\tau} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.9)$$

Let

$$P(H^{(J)}) = \{\alpha \in Q^-(J) \mid \dim H_{\alpha}^{(J)} \neq 0\} \ldots\ldots\ldots\ldots\ldots\ldots \quad (2.10)$$

and let $\{\tau_1, \tau_2, \tau_3, \cdots\}$, be an enumeration of the set $P(H^{(J)})$. Let $D(i) = DimH_{\tau_i}^{(J)}$.

Remark: The elements of $P(H^{(J)})$ can be determined by applying the following proposition, proved in Kac (1990).

Proposition 2.16: (Kang and Kim, 1997)

Let $\Lambda \in P_+$. Then $P(\Lambda) = W.\{\lambda \in P_+ \mid \lambda \text{ is nondegenerate with respect to } \Lambda\}$.

Now for $\tau \in Q^-(J)$, one can define

$$T^{(J)}(\tau) = \{n = (n_i)_{i \geq 1} \mid n_i \in \mathbb{Z}_{\geq 0}, \sum n_i \tau_i = \tau\}, \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.11)$$

which is the set of all partitions of τ into a sum of α_i's. For $n \in T^{(J)}(\alpha)$, use the notations $|n| = \sum n_i$ and $n! = \prod n_i!$.

Now, for $\tau \in Q^-(J)$, the *Witt partition function* $W^{(J)}(\tau)$ is defined as

$$W^{(J)}(\tau) = \sum_{n \in T^{(J)}(\tau)} \frac{(|n|-1)!}{n!} \prod D(i)^{n_i} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.12)$$

Now a closed form formula for the super dimension $Dim\ \mathfrak{g}_\alpha$ of the root space $\mathfrak{g}_\alpha\ (\alpha \in \Delta^-(J))$ is given by the following theorem. The proof is given in Kang and Kim(1997).

Theorem 2.17: *Let J be a finite subset of I^{re}. Then, for $\alpha \in \Delta^-(J)$, we have*

$$Dim\ \mathfrak{g}_\alpha = \sum_{d|\alpha} \frac{1}{d}\mu(d)W^{(J)}\left(\frac{\alpha}{d}\right)$$

$$= \sum_{d|\alpha} \frac{1}{d}\mu(d) \sum_{n\in T^{(J)}\left(\frac{\alpha}{d}\right)} \frac{(|n|-1)!}{n!} \prod D(i)^{n_i} \text{ ....(2.13)}$$

where μ is the classical Möbius function. Namely, for a natural number n, $\mu(n)$ is defined as follows:

$$\mu(n) = \begin{cases} 1 & for \quad n = 1, \\ (-1)^k & for \quad n = p_1 \cdots p_k \ (p_1, \cdots, p_k : distinct \quad primes), \\ 0 & if \ it \ is \ not \ square \ free \end{cases}$$

and, for a positive integer d, $d \mid \alpha$ denotes $\alpha = d\alpha$ for some $\alpha \in Q_-$, in which case $\alpha = \frac{\alpha}{d}$.

In the following sections 3.1 and 3.2, we find root supermultiplicities of Borcherds superalgebras which are the Extensions of Kac-Moody Algebras B_2 and B_3 (with multiplicity 1) and the corresponding combinatorial identities using Kang and Kim(1997).

III. Root supermultiplicities of some Borcherds superalgebras which are the Extensions of Kac Moody Algebras and the corresponding combinatorial identities

3.1 Superdimension formula and the corresponding combinatorial identity for the extended-hyperbolic Borcherds superalgebra which is an extension of B_2

Below, we find the dimension formulae and combinatorial identities for the Borcherds superalgebras which are extension of B_2. (for a same set $J \in \Pi^{re}$) by solving $T^{(J)}(\tau)$ in two different method.

Consider the extended-hyperbolic Borcherds superalgebra $\mathfrak{g} = \mathfrak{g}(A, \underline{m}, C)$ associated with the extended-hyperbolic Borcherds-Cartan super matrix,

$A = \begin{pmatrix} -k & -a & -b \\ -a & 2 & -1 \\ -b & -2 & 2 \end{pmatrix}$ and the corresponding coloring matrix C=$\begin{pmatrix} -1 & c_1 & c_2 \\ c_1^{-1} & 1 & c_3 \\ c_2^{-1} & c_3^{-1} & 1 \end{pmatrix}$ with $c_1, c_2, c_3 \in C^X$.

Let $I = \{1,2,3\}$ be the index set for the simple roots of $\mathfrak{g}$. Here α_1 is the imaginary odd simple root with multiplicity 1 and α_2, α_3 are the real even simple roots.

Let us consider the root $\alpha = k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3 \in Q$. We have $\theta(\alpha, \alpha) = (-1)^{k_1^2}$. Hence α is an even root(resp. odd root) if k_1 is even integer(resp. odd integer). Also $T = \{\alpha_1\}$ and the subset $F \subset T$ is either empty or $\{\alpha_1\}$. Take $J \subset \Pi^{re}$ as $J = \{3\}$. By Lemma 2.13, this implies that $W(J) = \{1, r_2\}$. From the equations(2.7) and (2.8), the homology space can be written as

$$H_1^{(J)} = V_J(-\alpha_1) \oplus V_J(-\alpha_2)$$

$$H_2^{(J)} = V_J(-\alpha_1 - (a+1)\alpha_2)$$

$$H_k^{(J)} = 0 \quad \forall k \geq 3$$

$$Therefore \quad H^{(J)} = H_1^{(J)} \Theta H_2^{(J)}$$

$$= V_J(-\alpha_1) \oplus V_J(-\alpha_2) \Theta V_J(-\alpha_1 - (a+1)\alpha_2)$$

with $DimH^{(J)}_{(1,0,0)} = -1;\ DimH^{(J)}_{(0,1,0)} = -1$

$DimH^{(J)}_{(1,1,1)} = -1;\ DimH^{(J)}_{(1,1,\ a+1)} = -1.$

We take $P(H^{(J)}) = \{(1,0,0),\ (0,1,0),\ (1,1,1),\ (1,1,\ a+1)\}.$

Let $\tau = \alpha = -p\alpha_1 - q\alpha_2 - u\alpha_3 \in Q_-$, with $(p,q,t) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$. Then by proposition.(2.16) we get $T^{(J)}(\tau) = \{(s_1, s_2, s_3, s_4) \mid s_1(1,0,0) + s_2(0,1,0) + s_3(1,1,1) + s_4(1,1,\ a+1) = (p,q,u)\}.$

This implies

$$s_1 + s_3 + s_4 = p$$
$$s_2 + s_3 + s_4 = q$$
$$s_3 + (a+1)s_4 = u$$

We have

$$s_1 = p - u + as_4$$
$$s_2 = q - u + as_4$$
$$s_3 = u - (a+1)s_4$$
$$s_4 = 0 \ to \ min\,(p, q, [\frac{u}{a+1}])$$

Applying s_1, s_2, s_3, s_4 in Witt partition formula(eqn.(2.12)), we have,

$$W^{(J)}(\tau) = \sum_{s_4=0}^{min\,(p,q,[\frac{u}{a+1}])} \frac{(p-u+q+as_4-1)!\,(-1)^{p-u+q+as_4}}{(p-q)!\,(q-u+as_4)!\,(u-(a+1)s_4)!\,s_4!} \quad \text{..........} \ \ ...(3.1.1)$$

From eqn(2.13), the dimension of $\mathfrak{g}_\alpha$ is

$$Dim\ \mathfrak{g}_\alpha = \sum_{d|\alpha} \frac{1}{d}\mu(d) W^{(J)}\left(\frac{\alpha}{d}\right)$$

$$= \sum_{d|\alpha} \frac{1}{d}\mu(d) \sum_{n \in T^{(J)}\left(\frac{\alpha}{d}\right)} \frac{(|n|-1)!}{n!} \prod D(i)^{n_i},$$

Substituting the value of $W^{(J)}(\tau)$ from eqn. (3.1.1) in the above dimension formula, we have,

$$Dim\ \mathfrak{g}_\alpha = \sum_{d|\alpha} \frac{1}{d}\mu(d) \sum_{s_4=0}^{min\,(\frac{p}{d},\frac{q}{d},[\frac{t}{d\,(a+1)}])} \frac{(\frac{p}{d} - \frac{u}{d} + \frac{q}{d} + \frac{a}{d}s_4 - 1)!\,(-1)^{p/d - u/d + q/d + as_4/d}}{(\frac{p}{d} - \frac{q}{d})!\,(\frac{q}{d} - \frac{u}{d} + \frac{a}{d}s_4)!\,(\frac{u}{d} - \frac{(a+1)}{d}s_4)!\,(s_4/d)!}. \quad \text{..........} \ ..(3.1.2)$$

If we solve the same $P(H^{(J)})$, using partition and substituting this partition in $T^{(J)}(\tau)$, we have

$$T^{(J)}(\tau) = \{(p-q), \phi_1 - u, u - (a+1)\phi_2, \phi_2\},$$

where ϕ_1 is the partition of `u' with parts $(1,\ a+1)$ of length 'u' and ϕ_2 is the partition of s_4 with parts upto $\min(p, q, [\frac{u}{a+1}])$. Applying $T^{(J)}(\tau)$ in Witt partition formula(eqn.(2.12)), we have

$$W^{(J)}(\tau) = \sum_{\phi_1, \phi_2 \in T^{(J)}(\tau)} \frac{(p-u+\phi_1-1)!\,(-1)^{p-u+\phi_1}}{(p-q)!\,(\phi_1-u)!\,(u-(a+1)\phi_2)!\,\phi_2!} \quad \text{..........} \ \text{........}(3\ .1.3)$$

From eqn.(2.13), the dimension of $\mathfrak{g}_\alpha$ is

$$Dim\ \mathfrak{g}_\alpha = \sum_{d|\alpha} \frac{1}{d}\mu(d) W^{(J)}\left(\frac{\alpha}{d}\right)$$

$$= \sum_{d|\alpha} \frac{1}{d}\mu(d) \sum_{n \in T^{(J)}\left(\frac{\alpha}{d}\right)} \frac{(|n|-1)!}{n!} \prod D(i)^{n_i},$$

Substituting the value of $W^{(J)}(\tau)$ from eqn. (3.1.2) in the above dimension formula, we have

$$Dim\ \mathfrak{g}_\alpha = \sum_{d|\alpha} \frac{1}{d}\mu(d) \sum_{n \in T^{(J)}\left(\frac{\alpha}{d}\right)} \sum_{\phi_1,\phi_2 \in T^{(J)}(\tau)} \frac{(\frac{p}{d} - \frac{u}{d} + \frac{\phi_1}{d} - 1)!\,(-1)^{p/d - u/d + \phi_1/d}}{(\frac{p}{d} - \frac{q}{d})!\,(\frac{\phi_1}{d} - \frac{u}{d})!\,(\frac{u}{d} - (a+1)(\phi_2/d))!\,(\phi_2/d)!}$$

Which gives another form of dimension of $\mathfrak{g}_\alpha$.

Applying the value of s_4 in (3.1.1), we get

$$W^{(J)}(\tau) = \sum_{s_4=0}^{\min(p,q,[\frac{u}{a+1}])} \frac{(p-u+q+as_4-1)!\,(-1)^{p-u+q+as_4}}{(p-q)!\,(q-u+as_4)!\,(u-(a+1)s_4)!\,s_4!}$$

$$= \frac{(p-u+q-1)!\,(-1)^{p-u+q}}{(p-q)!\,(q-u)!\,(u)!}$$

$$+ \frac{(p-u+q+a-1)!\,(-1)^{p-u+q+a}}{(p-q)!\,(q-u+a)!\,(u-(a+1))!1\ !} + \ldots\ldots.\ upto\ \ \min(p,q,[u/a+1])$$

$$= \sum_{\phi_1,\phi_2 \in T^{(J)}(\tau)} \frac{(p-u+\phi_1-1)!\,(-1)^{p-u+\phi_1}}{(p-q)!\,(\phi_1-u)!\,(u-(a+1)\phi_2)!\,\phi_2!}.$$

where ϕ_1 is the partition of `u' with parts $(1, a)$ of length '$u + as_4$' and ϕ_2 is the partition of s_4 with parts upto $\min(p,q,[\frac{u}{a+1}])$.

Hence, we get the following theorem.

Theorem 3.1.1: For the extended-hyperbolic Borcherds superalgebra $\mathfrak{g} = \mathfrak{g}(A, \underline{m}, C)$ associated with the extended-hyperbolic Borcherds-Cartan super matrix $A = \begin{pmatrix} -k & -a & -b \\ -a & 2 & -1 \\ -b & -2 & 2 \end{pmatrix}$ with charge $\underline{m} = \{1,1,1\}$, consider the root $\alpha = -p\alpha_1 - q\alpha_2 - u\alpha_3 \in Q_-$. Then the dimension of $\mathfrak{g}_\alpha$ is

$$Dim\ \mathfrak{g}_\alpha = \sum_{d|\alpha} \frac{1}{d}\mu(d) \sum_{s_4=0}^{\min(\frac{p}{d},\frac{q}{d},[\frac{t}{d(a+1)}])} \frac{(\frac{p}{d} - \frac{u}{d} + \frac{q}{d} + \frac{a}{d}s_4 - 1)!\,(-1)^{p/d - u/d + q/d + as_4/d}}{(\frac{p}{d} - \frac{q}{d})!\,(\frac{q}{d} - \frac{u}{d} + \frac{a}{d}s_4)!\,(\frac{u}{d} - \frac{(a+1)}{d}s_4)!\,(s_4/d)!}.$$

Moreover the following combinatorial identity holds:

$$\sum_{s_4=0}^{\min(p,q,[\frac{u}{a+1}])} \frac{(p-u+q+as_4-1)!\,(-1)^{p-u+q+as_4}}{(p-q)!\,(q-u+as_4)!\,(u-(a+1)s_4)!\,s_4!} = \sum_{\phi_1,\phi_2\in T^{(J)}(\tau)} \frac{(p-u+\phi_1-1)!\,(-1)^{p-u+\phi_1}}{(p-q)!\,(\phi_1-u)!\,(u-(a+1)\phi_2)!\,\phi_2!} \quad .....(3.1.4)$$

where ϕ_1 is the partition of `u' with parts $(1, a)$ of length '$u+as_4$' and ϕ_2 is the partition of s_4 with parts upto $\min(p,q,[\frac{u}{a+1}])$.

Example 3.1.2: For the Borcherds-Cartan super matrix $A = \begin{pmatrix} -k & -1 & -b \\ -1 & 2 & -1 \\ -b & -2 & 2 \end{pmatrix}$, consider a root $\alpha = \tau = (5,4,3)$ with a=1.

Substituting $\alpha = \tau = (5,4,3)$, $a = 1$ in eqn(3.1.1), we have

$$W^{(J)}(\tau) = \sum_{s_4=0}^{\min(p,q,[\frac{u}{a+1}])} \frac{(p-u+q+as_4-1)!\,(-1)^{p-u+q+as_4}}{(p-q)!\,(q-u+as_4)!\,(u-(a+1)s_4)!\,s_4!}$$

$$= \sum_{s_4=0}^{\min(5,4,[\frac{3}{2}])} \frac{(5-3+4+s_4-1)!\,(-1)^{5-3+4+s_4}}{(5-4)!\,(4-3+s_4)!\,(3-2s_4)!\,s_4!}$$

$$= \frac{5!\,(-1)^6}{1!1!3!} + \frac{7!\,(-1)^7}{1!2!1!1!} = 4.5 - 3.4.5.6.7 \quad = -2500.$$

Substituting $\alpha = \tau = (5,4,3)$, $a = 1$ in eqn(3.1.3), we have

$$W^{(J)}(\tau) = \sum_{\phi_1,\phi_2\in T^{(J)}(\tau)} \frac{(p-u+\phi_1-1)!\,(-1)^{p-u+\phi_1}}{(p-q)!\,(\phi_1-u)!\,(u-(a+1)\phi_2)!\,\phi_2!}$$

$$= \frac{5!\,(-1)^6}{1!1!3!} + \frac{7!\,(-1)^7}{1!2!1!1!} = -2500.$$

Hence the equality (3.1.4) holds.

3.2.Dimension Formula and combinatorial identity for the Borcherds superalgebra which is an extension of B_3

Here we are finding the superdimension Formula and combinatorial identity for the Borcherds superalgebra which is an extension of B_3 using the same $J \subset \Pi^{re}$ and solving $T^{(J)}(\tau)$ in two different ways.

Consider the extended-hyperbolic Borcherds superalgebra $\mathfrak{g} = \mathfrak{g}(A, \underline{m}, C)$ associated with the extended-hyperbolic Borcherds-Cartan super matrix $A = \begin{pmatrix} -k & -a & -b & -c \\ -a & 2 & -1 & 0 \\ -b & -1 & 2 & -1 \\ -c & 0 & -2 & 2 \end{pmatrix}$ and the corresponding coloring matrix C= $\begin{pmatrix} -1 & c_1 & c_2 & c_3 \\ c_1^{-1} & 1 & c_4 & c_5 \\ c_2^{-1} & c_4^{-1} & 1 & c_6 \\ c_3^{-1} & c_5^{-1} & c_6^{-1} & 1 \end{pmatrix}$ with $c_1, c_2, c_3, c_4 \in C^X$.

Let $I = \{1,2,3,4\}$ be the index set with charge $\underline{m} = \{1,1,1,1\}$.

Let us consider the root $\alpha = k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3 + k_4\alpha_4 \in Q$. We have $\theta(\alpha,\alpha) = (-1)^{k_1^2}$. Hence α is an even root(resp. odd root) if k_1 is even integer(resp. odd integer). Also $T = \{\alpha_1\}$ and the subset $F \subset T$ is either empty or $\{\alpha_1\}$. Take $J \subset \Pi^{re}$ as $J = \{2,3\}$. By Lemma 2.13, this implies that $W(J) = \{1, r_4\}$. From the equations (2.7) and (2.8),the homological space can be written as

$$H_1^{(J)} = V_J(1(\rho - \alpha_1) - \rho) \oplus V_J(r_4(\rho) - \rho)$$

$$= V_J(-\alpha_1) \oplus V_J(-\alpha_4)$$

$$H_2(J) = V_J(-\alpha_1 - (c+1)\alpha_4)$$

$$H_k^{(J)} = 0 \quad \forall k \geq 3$$

$$Therefore H^{(J)} = H_1^{(J)} = V_J(-\alpha_1) \oplus V_J(-\alpha_4)! V_J(-\alpha_1 - (c+1)\alpha_4)$$

with

$$DimH^{(J)}_{(1,0,0,0)} = -1; \quad DimH^{(J)}_{(1,1,0,0)} = -1; \quad DimH^{(J)}_{(1,1,1,0)} = -1;$$

$$DimH^{(J)}_{(1,1,1,1)} = -1; \quad DimH^{(J)}_{(1,1,1,\ c+1)} = -1$$

Hence we have

$$P(H^{(J)}) = \{(1,0,0,0), (1,1,0,0), (1,1,1,0), (1,1,1,1), (1,1,1, c+1)\}.$$

Let $\alpha = \tau = -p\alpha_1 - q\alpha_2 - u\alpha_3 - v\alpha_4 \in Q_-$, with $(p,q,u,v) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$. Then by proposition.(2.16), we get

$$T^{(J)}(\tau) = \{(s_1, s_2, s_3, s_4, s_5) \mid s_1(1,0,0,0) + s_2(1,1,0,0) + s_3(1,1,1,0)$$
$$+ s_4(1,1,1,1) + s_5(1,1,1, c+1) = (p,q,u,v)\}.$$

This implies
$$s_1 + s_2 + s_3 + s_4 = p$$
$$s_2 + s_3 + s_4 + s_5 = q$$
$$s_3 + s_4 + s_5 = u$$
$$s_4 + (c+1)s_5 = v.$$

We have
$$s_1 = p - q + v + cs_5$$
$$s_2 = q - u$$
$$s_3 = u - v - cs_5$$
$$s_4 = q - p - s_5$$
$$s_5 = 0 \ to \ min\ (p,q,u,[\frac{v}{c+1}]).$$

Applying s_1, s_2, s_3, s_4, s_5 in Witt partition formula(eqn.2.12), we have

$$W^{(J)}(\tau) = \sum_{s_5=0}^{min\ (p,q,u,[\frac{v}{c+1}])} \frac{(q-1)!\,(-1)^q}{(p-q+v+cs_5)!\,(q-u)!\,(u-v-cs_5)!\,(q-p-s_5)!\,s_5!} \quad \text{.......... ...(3.2.1)}$$

From equation(2.13), the dimension $\mathfrak{g}_\alpha$ is

$$Dim\ \mathfrak{g}_\alpha = \sum_{d|\alpha} \frac{1}{d}\mu(d)W^{(J)}\left(\frac{\alpha}{d}\right)$$

$$= \sum_{d|\alpha} \frac{1}{d}\mu(d) \sum_{n \in T^{(J)}\left(\frac{\alpha}{d}\right)} \frac{(|n|-1)!}{n!} \prod D(i)^{n_i},$$

Substituting the value of $W^{(J)}(\tau)$ from eqn. (3.3.1) in the above dimension formula, we have,

$$Dim\ \mathrm{g}_\alpha = \sum_{d|\alpha} \frac{1}{d}\mu(d) \sum_{\substack{s_5=0 \\ (u-v+s_5)\geq 0}}^{min\ (p/d,q/d,u/d,[\frac{v}{d(c+1)}])} \frac{(\frac{q}{d}-1)!\ (-1)^{q/d}}{(\frac{p}{d}-\frac{q}{d}+\frac{v}{d}+c\frac{s_5}{d})!\ (\frac{q}{d}-\frac{u}{d})!\ (\frac{u}{d}-\frac{v}{d}-c\frac{s_5}{d})!\ (\frac{q}{d}-\frac{p}{d}-\frac{s_5}{d})!\ \frac{s_5}{d}!}$$

If we solve the same $P(H^{(J)})$ using partition and substituting the partition, we have

$$T^{(J)}(\tau) = \{p-q+\phi_1, q-u, q-p-\phi_2, u-\phi_1, \phi_2\},$$

where ϕ_1 is partition of v with parts upto $(1, c+1)$ and of length v and ϕ_2 is the partition of s_5 with parts of s_5 of length s_5.

Applying $T^{(J)}(\tau)$ in Witt partition formula (eqn.(2.12))

$$W^{(J)}(\tau) = \sum_{\phi \in T^{(J)}(\tau)} \frac{(q-1)!\ (-1)^q}{(p-q+\phi_1)!\,(q-u)!\,(q-p-\phi_2)!\,(u-\phi_1)\,|\phi_2|!} \quad \text{..........\ ....(3 .2.2)}$$

From equation(2.13), the dimension g_α is

$$Dim\ \mathrm{g}_\alpha = \sum_{d|\alpha} \frac{1}{d}\mu(d) W^{(J)}\left(\frac{\alpha}{d}\right)$$

$$= \sum_{d|\alpha} \frac{1}{d}\mu(d) \sum_{n \in T^{(J)}\left(\frac{\alpha}{d}\right)} \frac{(|n|-1)!}{n!} \prod D(i)^{n_i},$$

Substituting the value of $W^{(J)}(\tau)$ from eqn. (3.2.2) in the above dimension formula, we have,

$$Dim\ \mathrm{g}_\alpha = \sum_{d|\alpha} \frac{1}{d}\mu(d) \sum_{\phi_1,\phi_2 \in T^{(J)}(\tau)} \frac{(q/d-1)!\ (-1)^{q/d}}{(p/d-q/d+\phi_1)!\,(q/d-u/d)!\,(q/d-p/d-\phi_2)!\,(u/d-\phi_1)\,|\phi_2|!}$$

Now consider the equation

$$\sum_{s_5=0}^{min\ (p,q,u,[\frac{v}{c+1}])} \frac{(q-1)!\ (-1)^q}{(p-q+v+cs_5)!\,(q-u)!\,(u-v-cs_5)!\,(q-p-s_5)!\,s_5!}.$$

$$= \frac{(q-1)!\ (-1)^q}{(p-q+v)!\ (q-u)!\ (u-v)!\ (q-p)!}$$

$$+ \frac{(q-1)!\ (-1)^q}{(p-q+v+c)!\ (q-u)!\ (u-v-c)!\ (q-p-1)!1!}$$

$$+ \frac{(q-1)!\ (-1)^q}{(p-q+v+2c)!\ (q-u)!\ (u-v-2c)!\ (q-p-2)!2!}$$

$$+ \frac{(q-1)!\ (-1)^q}{(p-q+v+3c)!\ (q-u)!\ (u-v-3c)!\ (q-p-3)!3!}$$

$$+ \ldots\ldots.\ upto \quad s_5 = min\ (p,q,u,[\frac{v}{c+1}]).$$

$$= \sum_{\phi_1,\phi_2 \in T^{(J)}(\tau)} \frac{(q-1)!\,(-1)^q}{(p-q+\phi_1)!\,(q-u)!\,(q-p-\phi_2)!\,(u-\phi_1)\,|\phi_2|!}$$

where ϕ_1 is partition of v with parts upto $(1, c+1)$ and of length v and ϕ_2 is the partition of s_5 with parts of s_5 of length s_5.

Hence we proved the following theorem.

Theorem 3.2.1: For the extended-hyperbolic Borcherds superalgebra $\mathfrak{g} = \mathfrak{g}(A, \underline{m}, C)$ associated with the extended-hyperbolic Borcherds-Cartan super matrix $A = \begin{pmatrix} -k & -a & -b & -c \\ -a & 2 & -1 & 0 \\ -b & -1 & 2 & -1 \\ -c & 0 & -2 & 2 \end{pmatrix}$ let us consider $\alpha = \tau = -p\alpha_1 - q\alpha_2 - u\alpha_3 - v\alpha_4 \in Q_-$. Then the dimension of $\mathfrak{g}_\alpha$ is

$$Dim\ \mathfrak{g}_\alpha = \sum_{d|\alpha} \frac{1}{d}\mu(d) \sum_{\substack{s_5=0 \\ (u-v+s_5)\geq 0}}^{\min(p/d,\,q/d,\,u/d,\,[\frac{v}{d(c+1)}])} \frac{(\frac{q}{d}-1)!\,(-1)^{q/d}}{(\frac{p}{d}-\frac{q}{d}+\frac{v}{d}+c\frac{s_5}{d})!\,(\frac{q}{d}-\frac{u}{d})!\,(\frac{u}{d}-\frac{v}{d}-c\frac{s_5}{d})!\,(\frac{q}{d}-\frac{p}{d}-\frac{s_5}{d})!\,\frac{s_5}{d}!}.$$

Moreover the following combinatorial identity holds:

$$\sum_{s_5=0}^{\min(p,q,u,[\frac{v}{c+1}])} \frac{(q-1)!\,(-1)^q}{(p-q+v+cs_5)!\,(q-u)!\,(u-v-cs_5)!\,(q-p-s_5)!\,s_5!} =$$

$$\sum_{\phi_1,\phi_2 \in T^{(J)}(\tau)} \frac{(q-1)!\,(-1)^q}{(p-q+\phi_1)!\,(q-u)!\,(q-p-\phi_2)!\,(u-\phi_1)\,|\phi_2|!} \quad \text{..........}\ .(3.2.3)$$

where ϕ_1 is partition of v with parts upto $(1, c)$ and of length v and ϕ_2 is the partition of s_5 with parts of s_5 of length s_5.

Example 3.2.2: For the Borcherds-Cartan super matrix $A = \begin{pmatrix} -k & -1 & -b & -c \\ -1 & 2 & -1 & 0 \\ -b & -1 & 2 & -1 \\ -c & 0 & -2 & 2 \end{pmatrix}$,

consider the root $\alpha = \tau = (7,6,4,3) = (p, q, u, v)$ with c=1. Applying in equation (3.2.1), we have

$$\sum_{s_5=0}^{\min(p,q,u,[\frac{v}{c+1}])} \frac{(q-1)!\,(-1)^q}{(p-q+v+cs_5)!\,(q-u)!\,(u-v-cs_5)!\,(q-p-s_5)!\,s_5!}$$

$$= \sum_{s_5=0}^{1} \frac{(6-1)!\,(-1)^6}{(7-6+3+s_5)!\,(6-4)!\,(4-3-s_5)!\,(6-7-s_5)!\,s_5!}$$

$$= \frac{5!\,(1)}{4!2!1!0!} + \frac{5!\,(1)}{5!2!0!0!1!}$$

$$= \frac{5}{2} + \frac{1}{2} = 3.$$

Consider the root $\tau = \alpha$ as $(7,6,4,3)$ with c=1. Applying in equation (3.2.2), we have

$$\sum_{\phi_1,\phi_2\in T^{(J)}(\tau)} \frac{(q-1)!\,(-1)^q}{(p-q+\phi_1)!\,(q-u)!\,(q-p-\phi_2)!\,(u-\phi_1)\,|\,\phi_2\,|!}$$

$$= \frac{(6-1)!\,(-1)^6}{(7-6+3)!\,(6-4)!\,(4-3)!\,(6-7)!0!} + \frac{(6-1)!\,(-1)^6}{(7-6+3+1)!\,(6-4)!\,(4-3-1)!\,(6-7)!1!}$$

$$= \frac{5}{2} + \frac{1}{2} = 3.$$

Hence the equality (3.2.3) holds.

Remark: In the above two sections 3.1 and 3.2., the identities (3.1.4) and (3.2.3) hold for any root, because we have derived the identities by simply solving the $T^{(J)}(\tau)$ in two different ways.

Remark: It is hoped that, in general, superdimensions of roots and the corresponding combinatorial identities for Borcherds Superalgebras which are extensions of all finite dimensional Kac-Moody algebras and superdimensions for all other categories can also be found out.

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