The International Journal Of Engineering And Science (IJES)
||Volume||2 ||Issue|| 5 ||Pages|| 65-75||2013||
ISSN(e): 2319 – 1813 ISSN(p): 2319 – 1805

In Shooting Pool, Part I: How Does The Cue Ball Go?

Ronald L. Huston
Mechanical EngineeringUniversity of Cincinnati
Cincinnati,

--ABSTRACT--
In shooting pool, as with any ball sport, seemingly minor nuances in the shot can result in significant changes in the subsequent ball dynamics. In addition to variation in shooting kinematics and kinetics, there are numerous popular notions about how to obtain desired results. In this first of a three-part series we address the title question by quantifying the interactive effects of: 1) the location of the point of impact between the cue stick and the cue ball; 2) the cue stick orientation; 3) the impulsive force magnitude; and 4) the friction between the ball and table surfaces. The analysis is based upon the solution of eight simultaneous equations obtained using classical mechanics principles. Various specific cases are then considered, illustrated, and discussed.

KEYWORDS:*pocket billiards, pool shooting, rolling/sliding spheres, impact on sphere dynamics*

Date Of Submission: 13 April 2013 ⟺ Date Of Publication: 28,May.2013

I. INTRODUCTION

In a recent conversation with a university administrator she stated that she was an expert pool player. When asked for the secret of her expertise she stated: “It’s all in the direction of the cue stick. The cue ball goes in the direction of the cue stick regardless of where the ball is struck by the stick.” Is this true? What is the effect, if any, of the cue stick not being in a plane of a vertical great circle of the ball?

The objectives of this and the subsequent two papers are to address these and related questions, such as: Where does the struck object ball go? And then: Where will the rebounding cue ball go?To answer these questions we (the writer and readers) consider first the induced dynamics of a ball struck with an impulsive force applied at an arbitrary point on the ball surface with an arbitrary magnitude and direction. We assume ideal geometry (a spherical ball in point contact with a flat horizontal table surface) but we restrict the analysis to cue stick (herein called the “cue”) forces directed so that the ball remains on the table. We include the effect of friction between the ball and the table.The sport of pool (or “pocket billiards”) is a centuries-old game. Numerous articles and books can be found discussing various game rules, game strategy, shooter posture, and skill development. But only a few of these writings provide an analytical discussion of ball dynamics. Perhaps the most noteworthy of the analytical discussions is that provided by Marlow [1]. The writings of Mosconi [2], of Mizerak, Panozzo, and Fels [3], of Pejcic and Meyer [4], and of Alciatore [5] are also useful. In this paper we seek to present a somewhat more focused analysis of cue/cue ball dynamics leading to an elementary algorithm for studying subsequent collisions and movements.

The balance of the paper is divided into five sections with the first of these providing the terminology used in the sequel, the basic assumptions, and the definition of the problem being solved. The next section presents a kinetic, kinematic, and dynamic analysis resulting in the governing equations of motion. These equations are solved in the subsequent section. The solution algorithm is then illustrated via a series of special problems in the penultimate section. The final section presents a brief discussion and concluding remarks.

II. TERMINOLOGY, ASSUMPTIONS, AND PROBLEM DEFINITION

Consider a sphere B (the “cue ball”) initially at rest on a flat horizontal surface S (the “table”) as in Fig. 1. Let G be the geometric and mass center of B. Let C be the contact point of B with S and let P be that point on the surface of B where an impulsive force **F** from the cue stick (the “cue”) is applied. Let X, Y, Z be a Cartesian axis system fixed relative to S, with the Z-axis, along CG, being normal to S. Let the origin O of the axis system initially coincide with the mass center G of B. Let $\mathbf{N}_X$, $\mathbf{N}_Y$, and $\mathbf{N}_Z$ be unit vectors parallel to X, Y, and Z.

Let r and m be the radius and mass of B. Let μ be the coefficient of friction between B and S. Let (x,y,z) be the X,Y,Z coordinates of P and let F_X, F_Y, and F_Z be the $\mathbf{N}_X$, $\mathbf{N}_Y$, and $\mathbf{N}_Z$ components of **F**. Finally, let N be the magnitude of the vertical (Z-axis, or $\mathbf{N}_Z$) component of the force exerted by S on B at C.

The given terminology and definitions imply the following assumptions:

[1] B is a rigid, homogeneous sphere.
[2] With S being a flat, horizontal surface, there is only point contact (at C) between B and S.
[3] The force system exerted by S on B at C may be represented by a single force **C** passing through C.
[4] If the velocity $\mathbf{V}^C$ of C relative to S is not zero, the magnitude of the horizontal component of **C** is simply μ N and its direction is opposite to that of $\mathbf{V}^C$.

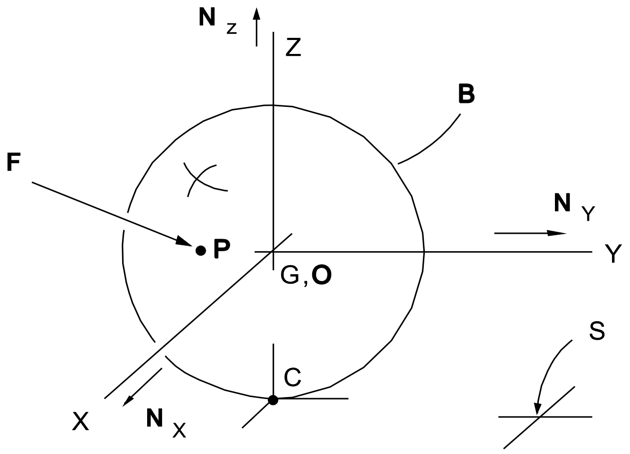

Fig. 1A sphere B representing a cue ball and a pool table S

[1] There is no slippage of the cue on the surface of S.
[2] During and following the impulse, B and S remain in contact.
[3] During the short impulse time there is relatively little change in the position and orientation of B, but there are significant changes in the velociyt$\mathbf{V}^G$of G and/or the angular velocity $\boldsymbol{\omega}^B$ of B.

With this notation and terminology, the fundamental problem to be solved is: Given (F_X, F_Y, F_Z), (x,y,z), m, r, and μ, determine the immediate $\mathbf{N}_X$, $\mathbf{N}_Y$, and $\mathbf{N}_Z$ components of the velocity $\mathbf{V}^G$and the angular velocity $\boldsymbol{\omega}^B$ (both relative to S).

III. Dynamic Analysis

To solve the problem we can use the foregoing definitions and assumptions to readily obtain the governing equations of motion. To this end, consider a free-body diagram of B as in Fig. 2, where the applied ("active") forces are: **F**, **W**, and **C** representing the cue, the weight, and the contact forces by S on B. Correspondingly the inertia ("passive") forces are represented by the force $\mathbf{F}^*$ passing through the mass center G and a couple with torque $\mathbf{T}^*$.

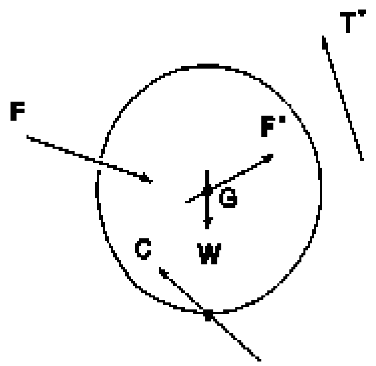

Fig. 2A free-body diagram of the ball B

In terms of the unit vectors of Fig. 1 the active forces may be expressed as:

$$\mathbf{F} = F_X\mathbf{N}_X + F_Y\mathbf{N}_Y + F_Z\mathbf{N}_Z \tag{1}$$
$$\mathbf{W} = -mg\mathbf{N}_Z \tag{2}$$
$$\mathbf{C} = C_X\mathbf{N}_X + C_Y\mathbf{N}_Y + N_Z\mathbf{N}_Z \tag{3}$$

Similarly, using d'Alembert's principle [6,7,8] we can express the inertia force $\mathbf{F}^*$ and torque $\mathbf{T}^*$ as:

$$\mathbf{F}^* = -m\mathbf{a}^G = -m(a_X\mathbf{N}_X + a_Y\mathbf{N}_Y + a_Z\mathbf{N}_Z) \tag{4}$$

and

$$\mathbf{T}^* = -\mathbf{I} \bullet \boldsymbol{\alpha}^B - \boldsymbol{\omega}^B \times (\mathbf{I} \bullet \omega^B) = -\mathbf{I}(\alpha_X \mathbf{N}_X + \alpha_Y \mathbf{N}_Y + \alpha_Z \mathbf{N}_Z) \tag{5}$$

Where g is the gravity acceleration; $\mathbf{a}^G$ is the acceleration of G relative to the table S, with X, Y, Z components (a_X, a_Y, a_Z); $\mathbf{I}$ is the central inertia dyadic [6,7,8]; $\boldsymbol{\omega}^B$ is the angular velocity of B in S (as before); and I is the moment of inertia of B, about a diameter, given by: $(2/5)mr^2$.

In (5), due to the symmetry of the sphere, $\mathbf{I}$ is simply I multiplied by the identity dyadic. Consequently, $\mathbf{I} \bullet \boldsymbol{\omega}^B$ is simply $I\boldsymbol{\omega}^B$ and thus the second term in the second expression of (5) vanishes.

By setting the resultant of the forces in Fig. 2 equal to zero, and by setting the moments of the forces about G equal to zero, we have:

$$\mathbf{F} + \mathbf{W} + \mathbf{C} + \mathbf{F}^* = 0 \tag{6}$$

and

$$\mathbf{p} \times \mathbf{F} + (-r\mathbf{N}_3) \times \mathbf{C} + \mathbf{T}^* = 0 \tag{7}$$

Where $\mathbf{p}$ is a position vector locating P relative to G.

Upon impact from the cue, the velocity of the contact point C generally will not be zero. Therefore to develop the analysis and specifically to develop the terms of (5) and (6) it is helpful to initially consider: $\mathbf{V}^C \neq 0$ and then let $\mathbf{V}^C=0$ be a special case.

In this regard, consider an overview of B depicting $\mathbf{V}^C$ as in Fig. 3, where θ defines the inclination of $\mathbf{V}^C$ relative to $\mathbf{N}_X$. Let $\mathbf{n}_C$ be a unit vector parallel to $\mathbf{V}_C$ with the same sense as $\mathbf{V}^C$. Then by the Coulomb friction law the X and Y components of the contact force $\mathbf{C}$ may be expressed as:

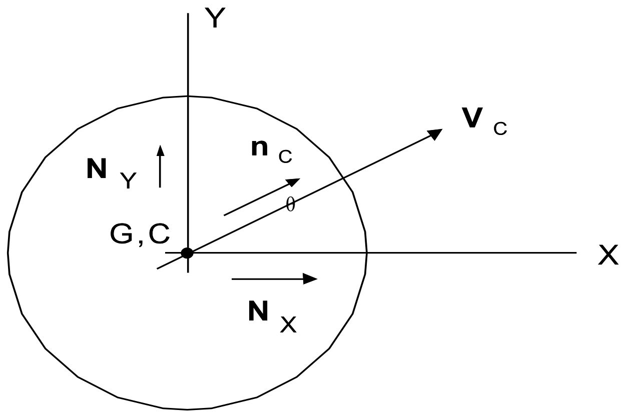

Fig. 3 A Top/Down View of B.

$$C_X = -\mu N \cos\theta \quad \text{and} \quad C_Y = -\mu N \sin\theta \tag{8}$$

By substituting from (1) through (5) together with (8) into (6) and (7) and by projecting the resulting vector equations along X, Y, and Z we immediately obtain six scalar expressions:

$$F_X - \mu N \cos\theta - m a_x = 0 \tag{9}$$
$$F_Y - \mu N \sin\theta - m a_Y = 0 \tag{10}$$
$$F_Z - mg + N = 0 \tag{11}$$
$$yF_Z - zF_Y - \mu r N \sin\theta - (2/5)mr^2\alpha_X = 0 \tag{12}$$

$$zF_X - xF_Z + \mu rN\cos\theta - (2/5)mr^2\alpha_Y = 0 \tag{13}$$

$$xF_Y - yF_X - (2/5)mr^2\alpha_Z = 0 \tag{14}$$

Next, since C and G are both fixed in B, we have the kinematic relation:

$$\mathbf{V}^G = \mathbf{V}^C + \boldsymbol{\omega}^B \times r\mathbf{n}_z \tag{15}$$

Where $\mathbf{n}_z$ is a member of the unit vector set: $\mathbf{n}_x$,$\mathbf{n}_y$, and $\mathbf{n}_z$, fixed in B, and initially parallel to $\mathbf{N}_X$, $\mathbf{N}_Y$, and $\mathbf{N}_Z$ respectively. Since there is little change in the orientation of B during the short time interval of the cue impact, $\mathbf{n}_z$ remains nearly parallel to $\mathbf{N}_Z$ during the impact time. Then by expressing (15) in component form we have:

$$V_X^G = V_C\cos\theta + r\omega_Y \tag{16}$$

and

$$V_Y^G = V_C\sin\theta - r\omega_X \tag{17}$$

Where V_X^G, V_Y^G, ω_X and ω_Y are $\mathbf{N}_X$ and $\mathbf{N}_Y$ components of $\mathbf{V}^G$ and $\boldsymbol{\omega}^B$, and where V_C is the magnitude of $\mathbf{V}^C$.

The assumption that: there is neither little position nor orientation change of B during the cue impact, is equivalent to the impact time t^* being small. Indeed, tests show that the impact time may be as low as 10 ms (0.01 sec).

The short impact time enables us to simplify the dynamical equations by integrating through the time interval. To this end, it is helpful to make the notational definitions:

$$\begin{aligned} &V_X^G\left(t^*\right) = V_X^*\ ,\quad V_Y^G\left(t^*\right) = V_Y^*\ ,\quad V^C\left(t^*\right) = V_C^* \\ &\omega_X\left(t^*\right) = \omega_X^*\ ,\quad \omega_Y\left(t^*\right) = \omega_Y^*\ ,\quad \omega_Z\left(t^*\right) = \omega_Z^* \end{aligned} \tag{18}$$

Similarly, it is helpful to define the following impulses:

$$\hat{F}_X = \int_0^{t^*} F_X\,dt\ ,\quad \hat{F}_Y = \int_0^{t^*} F_Y\,dt\ ,\quad \hat{F}_Z = \int_0^{t^*} F_Z\,dt\ ,\quad \hat{N} = \int_0^{t^*} N\,dt \tag{19}$$

(Observe that the units of these impulses are: force-time.)

Then we have the following results:

$$\begin{aligned} &V_X^* = \int_0^{t^*} a_X\,dt\ ,\quad V_Y^* = \int_0^{t^*} a_Y\,dt\ ,\quad \omega_X^* = \int_0^{t^*} \alpha_X\,dt \\ &\omega_Y^* = \int_0^{t^*} \alpha_Y\,dt\ ,\quad \omega_Z^* = \int_0^{t^*} \alpha_Z\,dt \end{aligned} \tag{20}$$

By integrating through t^* in (9) through (14) we have:

$$\hat{F}_X - \mu\hat{N}\cos\theta - mV_X^* = 0 \tag{21}$$

$$\hat{F}_Y - \mu\hat{N}\sin\theta - mV_Y^* = 0 \tag{22}$$

$$\hat{F}_Z - mgt^* + \hat{N} = 0 \tag{23}$$

$$y\hat{F}_Z - z\hat{F}_Y - \mu r\hat{N}\sin\theta - (2/5)mr^2\omega_X^* = 0 \tag{24}$$

$$z\hat{F}_X - x\hat{F}_Z + \mu r\hat{N}\cos\theta - (2/5)mr^2\omega_Y^* = 0 \tag{25}$$

$$x\hat{F}_Y - y\hat{F}_X - (2/5)mr^2\omega_Z^* = 0 \tag{26}$$

Also with this notation (16) and (17) become:

$$V_X^* = V_C^* \cos\theta + r\omega_Y^* \tag{27}$$

$$V_Y^* = V_C^* \sin\theta - r\omega_X^* \tag{28}$$

IV. SOLUTION ALGORITHM

(21) and (28) form a system of eight algebraic equations for the eight unknowns: V_X^*, V_Y^*, $\hat{N}$, θ, ω_X^*, ω_Y^*, ω_Z^*, and V_C^*. [The other parameters ($\hat{F}_X$, $\hat{F}_Y$, $\hat{F}_Z$, μ, m, g, r, x, y, z) are assumed to be known.] Since the unknowns are distributed sparsely across the equations, the solution of the system may readily be obtained using Gauss elimination:

Specifically, from (23) and (26) $\hat{N}$ and ω_Z^* are:

$$\hat{N} = mgt^* - \hat{F}_Z \tag{29}$$

and

$$\omega_Z^* = \left(5/2mr^2\right)\left(x\hat{F}_Y - y\hat{F}_X\right) \tag{30}$$

Next, by substituting for $\hat{N}$ in (21) and (22), and by substituting for V_X^* and V_Y^* from (27) and (28), (21) and (22) may be written as:

$$mr^2\omega_Y^* = r\hat{F}_X - \mu\left(mgt^* - \hat{F}_Z\right)r\cos\theta - mrV_C^*\cos\theta \tag{31}$$

and

$$mr^2\omega_X^* = -r\hat{F}_Y + \mu\left(mgt^* - \hat{F}_Z\right)r\sin\theta + mrV_C^*\sin\theta \tag{32}$$

Then by substituting in turn for $mr^2\omega_X^*$, $mr^*\omega_Y^*$, and $\hat{N}$ in (24) and (25) and rearranging terms we have:

$$r\left[(7/5)\mu\left(mgt^* - \hat{F}_Z\right) + (2/5)mV_C^*\right]\sin\theta = y\hat{F}_Z - \left[z - (2r/5)\right]\hat{F}_Y \tag{33}$$

and

$$r\left[(7/5)\mu\left(mgt^* - \hat{F}_Z\right) + (2/5)mV_C^*\right]\cos\theta = x\hat{F}_Z - \left[z - (2r/5)\right]\hat{F}_X \tag{34}$$

Finally, by dividing these last two equations we obtain the relatively simple expression for determining θ:

$$\tan\theta = \frac{y\hat{F}_Z - \left[z - (2r/5)\right]\hat{F}_Y}{x\hat{F}_Z - \left[z - (2r/5)\right]\hat{F}_X} \tag{35}$$

Once θ is known we can immediately determine the remaining unknowns by back substitution. That is, knowing θ we can find V_C^* from either (33) or (34). (31) and (32) then produce ω_X^* and ω_Y^*. Finally, (27) and (28) provide V_X^* and V_Y^*.

There is, however, a note of caution: While (35) determines θ the value is only accurate to within a multiple of π. Since $\tan\theta = \tan(\theta \pm n\pi)$ we need to select the value of the arctangent function appropriate for the given physical conditions (as in the following algorithm). It is easy to develop the foregoing procedure into a relatively simple seven-step algorithm:

1) Form the functions: f_x and f_y as:

$$f_x = x\hat{F}_Z - \left[z - (2r/5)\right]\hat{F}_X \tag{36}$$

and

$$f_y = y\hat{F}_Z - \left[z - (2r/5)\right]\hat{F}_Y \tag{37}$$

2) Form f defined as:

$$f = \left(f_x^2 + f_y^2\right)^{1/2} \qquad f > 0 \tag{38}$$

3) Determine $\sin\theta$ and $\cos\theta$ as:

$$\sin\theta = f_y / f \quad \text{and} \quad \cos\theta = f_x / f \tag{39}$$

Observe in (33) and (34) that the term: $r\left[(7/5)\mu\left(mgt^* - \hat{F}_Z\right) + (2/5)mV_C^*\right]$ is positive and equal to f. (It is positive since $\hat{F}_Z$ is negative, for otherwise the ball would not remain in contact with the table surface, and V_C^* being the magnitude of V^C at t^* is positive.) Note further that the term: $mgt*$ is likely to be negligible since t^* is very small.

4) Find θ using (39). That is,

$$\theta = \tan^{-1} f_y / f_x \tag{40}$$

The quadrant of θ is determined as follows: Observe that since $f > 0$, we have:

$$\text{sgn}(\sin\theta) = \text{sgn}\, f_y \quad \text{and} \quad \text{sgn}(\cos\theta) = \text{sgn}\, f_x \tag{41}$$

Then knowing the signs of $\sin\theta$ and $\cos\theta$ the quadrant of θ is immediately known from Table 1.

Table 1. Quadrant of Contact Point Velocity V^C Direction θ

Quadrant of θ	I $(0 \le \theta \le \pi/2)$	II $(\pi/2 \le \theta \le \pi)$	III $(\pi \le \theta \le 3\pi/2)$	IV $(3\pi/2 \le \theta \le 2\pi)$
$\sin\theta$	+	+	-	-
$\cos\theta$	+	-	-	+

5) Determine V_X^*, V_Y^*, and V_C^* using the expressions:

$$V_X^* = \left(\hat{F}_X / m\right) + \left(\mu\hat{F}_Z / m\right)\cos\theta - \mu g t^* \cos\theta \tag{42}$$

$$V_Y^* = \left(\hat{F}_Y / m\right) + \left(\mu\hat{F}_Z / m\right)\sin\theta - \mu g t^* \sin\theta \tag{43}$$

$$V_C^* = (5f / 2rm) + (7/2)(\mu / m)\hat{F}_Z - (7/2)\mu g t^* \tag{44}$$

{See (21), (22), and (29) and note that $f = r\left[(7/5)\mu\left(mgt^* - \hat{F}_Z\right) + (2/5)mV_C^*\right]$.}

6) Determine ω_X^*, ω_Y^*, and ω_Z^* using the expressions:

$$\omega_X^* = \left(V_C^* \sin\theta - V_Y^*\right) / r \tag{45}$$

$$\omega_Y^* = \left(-V_C^* \cos\theta + V_X^*\right) / r \tag{46}$$

$$\omega_Z^* = \left(x\hat{F}_Y - y\hat{F}_X\right)\left(5 / 2mr^2\right) \tag{47}$$

[See (26), (27), and (28).]

7) Determine $\hat{N}$ as:

$$\hat{N} = mgt^* - \hat{F}_Z \tag{48}$$

[See (23).]

V. ILLUSTRATIONS AND APPLICATIONS

In this section we consider a series of fundamental cue orientations and ball impact points. The objective is twofold; 1) to illustrate the utility of the algorithm and 2) to provide an answer to the title question.

Case 1.A Straight Shot

Consider first a horizontal cue directed along a diameter of the ball as represented in Fig. 4, which also provides a free-body diagram for the active (applied) forces on the ball.

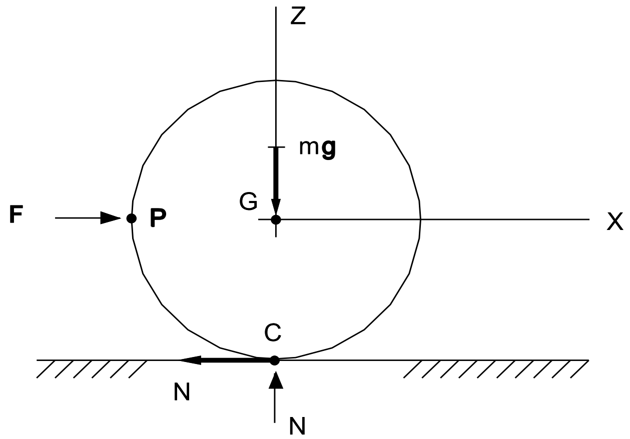

Fig. 4.A center-directed horizontal cue.

In this simple case the components of the impact force **F** and the coordinates of the point of impact P are: (F,0,0) and (-r,0,0).
Using this data, (36), (37), and (38) provide f_x, f_y, and f as:

$$f_x = (2r/5)\hat{F} \quad , \quad f_y = 0 \quad , \quad f = (2r/5)\hat{F} \tag{49}$$

where $\hat{F}$ is the impulse of F [see (19)]. (39), (40), and (41) then show that θ is zero, and consequently (42), (43), and (44) provide V_X^* , V_Y^* , and V_C^* as:

$$V_X^* = \hat{F}/m - \mu g t^* \; , \quad V_Y^* = 0 \; , \quad V_C^* = \hat{F}/m - (7/2)\mu g t^* \tag{50}$$

Then from (45), (46), and (47), ω_X^* , ω_Y^* , and ω_Z^* are:

$$\omega_X^* = 0 \; , \quad \omega_Y^* = (5/2)\mu g t / r \; , \quad \omega_Z^* = 0 \tag{51}$$

Finally, from (48) $\hat{N}$ is:

$$\hat{N} = m g t^* \tag{52}$$

Observe that, as expected, the ball simply moves on a straight line in the direction of the cue. It is sliding, but still rotating about the Y-axis, due to the ball/table friction.

Case 2.*A down-angled shot in a central ball plane.*
Consider next the configuration of a ball struck with a downward directed cue in a central (great circle) vertical plane as represented in Fig. 5, where the action forces are also shown.

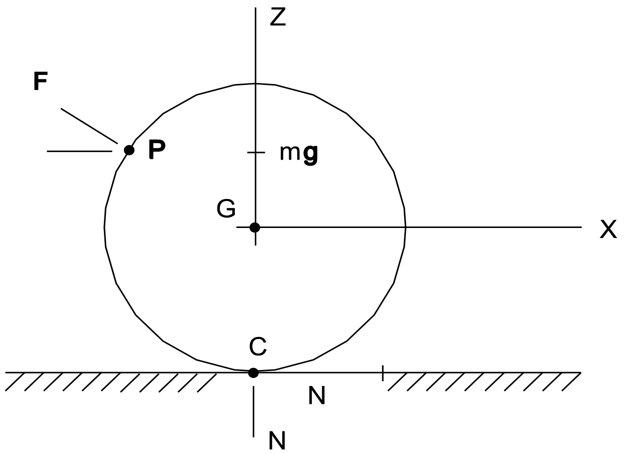

Fig. 5.A down-directed cue in a central ball plane.

Observe in this case that the direction of the friction force (μN) may be to the left (-X), or to the right (+X), depending upon the direction of the velocity of the contact point C $\left(V_C^*\right)$.

Here the components of **F** and the coordinates of P are: $(F\cos\alpha,\ 0,\ -f\sin\alpha)$ and (x, 0, z) where x is negative. (Note that $x^2 + z^2 = r^2$.)

With this data, f_x, f_y, and f become [(36) to (38)]:

$$f_x = -x\hat{F}\sin\alpha - \left[z - (2r/5)\right]\hat{F}\cos\alpha\ ,\quad f_y = 0\ ,\quad f = \left|f_x\right| \tag{53}$$

where, as before , $\hat{F}$ is the impulse of **F.**

Observe that f_x may be either positive or negative. When f_x is positive, $\theta = 0$; when f_x is negative, $\theta = \pi$.

Consider first $\theta = 0$: In this instance C moves in the positive X-direction (or is zero) and the friction force is directed in the negative X-direction as in Fig. 5. (42), (43), and (44) then provide V_X^*, V_Y^*, and V_C^* as:

$$V_X^* = \left(\hat{F}/m\right)(\cos\alpha - \mu\sin\alpha) - \mu g t^* \tag{54}$$

$$V_Y^* = 0 \tag{55}$$

and

$$V_C^* = \left(\hat{F}/m\right)\left\{-\left[(5x/2r) + (7\mu/2)\right]\sin\alpha + \left[1 - (5z/2r)\right]\cos\alpha\right\} - (7/2)\mu g t^* \tag{56}$$

Finally, (45) to (48) ω_X^*, ω_Y^*, ω_Z^*, and $\hat{N}$ are:

$$\omega_X^* = \omega_Z^* = 0 \tag{57}$$

$$\omega_Y^* = \left(5\hat{F}/2mr\right)\left\{\left[(x/r) + \mu\right]\sin\alpha + (z/r)\cos\alpha\right\} + \mu(g/r)t^* \tag{58}$$

and

$$\hat{N} = mgt^* + \hat{F}\sin\alpha \tag{59}$$

As noted earlier the terms with the factor t^* are likely to be insignificant, due to the short impact time. With this assumption, (54), (56), and (58) can provide insight about two special subcases: 1) zero speed of the ball center G, and 2) zero speed of the contact point C.

From (54), with $-\mu g t^*$ being neglected, we see that V_X^* is zero if:

$$\tan\alpha = 1/\mu \tag{60}$$

Thus, for small μ a relatively large cue angle is needed to keep the ball in place immediately after impact.

Next, if v_C^* is zero, the ball undergoes "rolling" [6, 7, 8] – that is, there is no sliding relative to the table. From (56) (with $\mu g t^*$ being neglected) we see that this can occur in several ways depending upon the values of x, z, and α. To briefly explore this, consider a horizontal cue, that is, $\alpha = 0$: (56) then produces the simple "rolling condition":

$$1 - (5z/2r) = 0 \quad \text{or} \quad z = (2/5)r \tag{61}$$

(58) then becomes:

$$\omega_Y^* = \hat{F} / mr \tag{62}$$

The point of impact represented by z = (2/5)r may be regarded as a "center of percussion."
Consider next $\theta = \pi$: In this instance f_x is negative. That is,

$$-x\hat{F}\sin\alpha - \left[z - (2r/5)\right]\hat{F}\cos\alpha < 0 \quad \text{or} \quad \tan\alpha < \left[z - (2r/5)\right]/(-x) \tag{63}$$

Note that x is negative (see Fig. 3). For a horizontal cue $(\alpha = 0)$, z must exceed: 2r/5. This in turn means that the point of impact must be above the center of percussion. Alternatively, if, say, $z = -x = \left(\sqrt{2}/2\right)r,\ \alpha$ must be less than 23 deg. In either event when $\theta = \pi$, the contact point C is moving in the negative X direction. The post-impact velocities are then routinely obtained using (42) to (48).

Case 3.*A slight error in cue alignment with the point of impact*
In the foregoing cases the cue was in the place of a great circle of the ball. The ball responded with its center G moving and remaining in that great circle plane. That is, the ball center moves in the direction of the cue. But what if the cue is not in a vertical great circle plane? We discuss that occurrence in this case.

Consider the ball being struck by the cue along the horizontal great circle, but a small distance η to the left of the vertical great circle, as represented in the three views of Fig. 6. Let the cue force **F** be directed parallel to the X-Z plane and downward at angle α as shown.

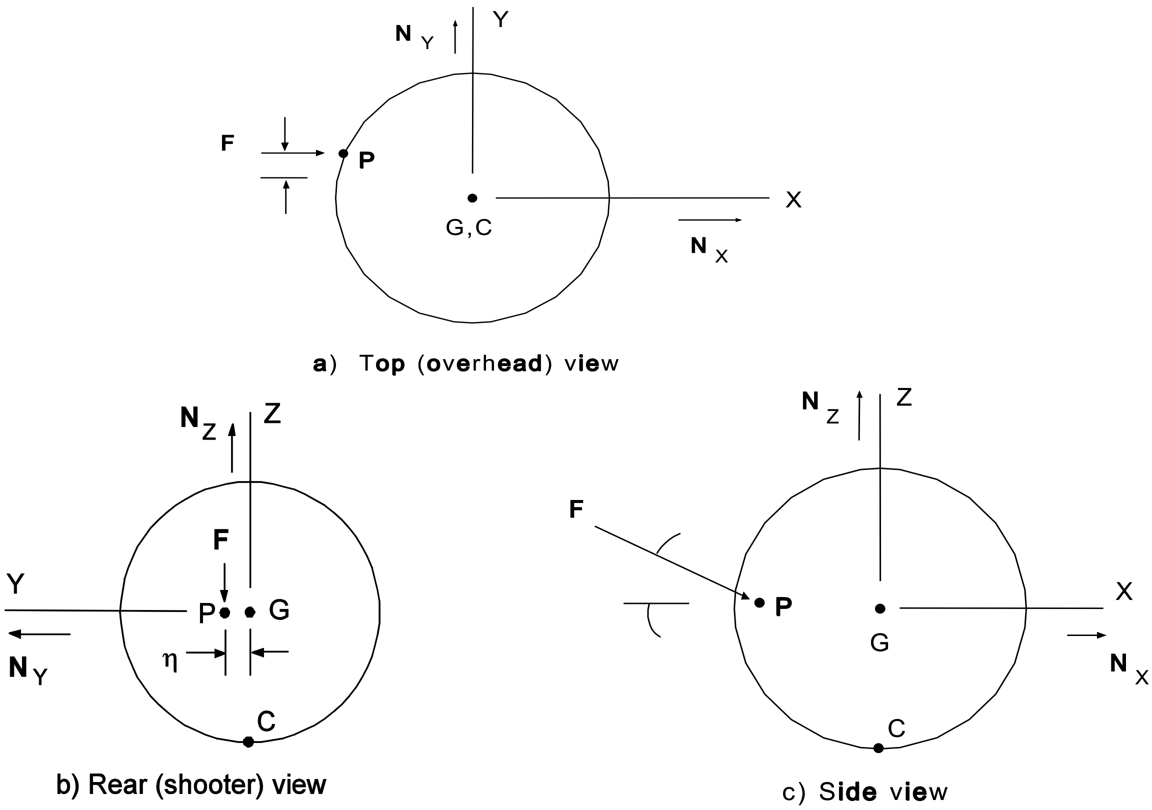

Fig. 6A ball struck slightly away from a great circle plane.

In this configuration the $\mathbf{N}_X$,$\mathbf{N}_Y$, $\mathbf{N}_Z$ components of the impulse of **F** and the X, Y, Z coordinates of the point of impact of P are:

$$\left(\hat{F}_X,\hat{F}_Y,\hat{F}_Z\right)=\left(\hat{F}\cos\alpha,0,-\hat{F}\sin\alpha\right)\quad\text{and}\quad(x,y,z)=\left[-\left(r^2-\eta^2\right)^{1/2},\eta,0\right] \tag{64}$$

If for simplicity both η and α are considered to be small, (64) may be written as:

$$\left(\hat{F}_X,\hat{F}_Y,\hat{F}_Z\right)=\left(\hat{F},0,-\hat{F}\alpha\right)\quad\text{and}\quad(x,y,z)=-(r,\eta,0) \tag{65}$$

With the data of (65), the algorithm equations of the foregoing section immediately provide the post-impact force functions and ball kinematics. Neglecting higher order terms, the results are:

$$f_X=r\left[(2/5)+\alpha\right]\hat{F},\quad f_Y=-\eta\alpha\hat{F},\quad f=r\left[(2/5)+\alpha\right]\hat{F} \tag{66}$$

$$\theta=-(5/2r)\eta\alpha \tag{67}$$

$$V_X^*=(1-\mu\alpha)\left(\hat{F}/m\right),\quad V_Y^*=(5/2)(\eta/r)\mu\alpha^2\left(\hat{F}/m\right) \tag{68}$$

$$V_C^*=\left\{1+\left[(5/2)-(7/2)\mu\right]\alpha\right\}\left(\hat{F}/m\right) \tag{69}$$

$$\omega_X^*=-(5/2)\left(\hat{F}/m\right)\eta\alpha/r^2,\quad \omega_Y^*=-(5/2)\left(\hat{F}/mr\right)(1-\mu)\alpha,\quad \omega_Z^*=-(5/2)\eta\left(\hat{F}/mr^2\right) \tag{70}$$

Observe in (68) that V_Y^* , although quite small, is not zero. That is, the ball center G moves away from the vertical plane containing the cue. Of perhaps greater significance is the finding of (67) that the contact point C also moves away from the vertical cue plane, but on the other side. The angle between the directions of the velocities of G and C is seen to be approximately: $(5/2)(\eta/r)\alpha$.

It has long been established that when the ball is sliding on the table surface and when the velocities of the center and the contact point are in different directions, the ball center moves along a parabolic curve [9]. Therefore, in this case the cue ball does not go in the direction of the cue.

Case 4. *A textbook example*

To illustrate the immediate foregoing remarks T. R. Kane in 1968 posed the problem: Given the impact force components and the impact point components as:

$$\left(\hat{F}_X,\hat{F}_Y,\hat{F}_Z\right)=\left(0,\hat{F}\sqrt{2}/2,-\hat{F}\sqrt{2}/2\right)\quad\text{and}\quad(x,y,z)=\left(r\sqrt{2}/2,0,r\sqrt{2}/2\right) \tag{71}$$

Find the cosine of the angle between the post-impact center and contact point velocities, with $\mu=0.1$ [6].

With the given data, (36), (37), and (38) provide the post-impact force functions as:

$$f_x=-r\hat{F}/2=-0.5r\hat{F},\quad f_y=-\left(5-2\sqrt{2}\right)r\hat{F}/10=-0.217r\hat{F}$$

$$\text{and}\quad f=r\hat{F}\left[\left(29-10\sqrt{2}\right)/50\right]^{1/2}=0.545r\hat{F} \tag{72}$$

Then from (39) $\sin\theta$ and $\cos\theta$ are:

$$\sin\theta=f_y/f=-0.398\quad\text{and}\quad\cos\theta=f_x/f=-0.917 \tag{73}$$

With both $\sin\theta$ and $\cos\theta$ being negative, Table 1 shows θ to be in the third quandrant. Specifically,

$$\tan\theta=0.434\quad\text{or}\quad\theta=203.46\ \text{deg} \tag{74}$$

Next, (42) and (43) show V_X^* and V_Y^* to be (neglecting the small terms involving $\mu g t^*$):

$$V_X^*=0.0648\hat{F}/m\quad\text{and}\quad V_Y^*=0.735\hat{F}/m \tag{75}$$

Finally, let $\mathbf{n}_G$ be a unit vector parallel to the center velocity and, as before, let $\mathbf{n}_C$ be a unit vector parallel to the contact point velocity. Then from (73) and (75) $\mathbf{n}_G$ and $\mathbf{n}_C$ are:

$$\mathbf{n}_G = 0.0878\,\mathbf{N}_X + 0.9961\,\mathbf{N}_Y \quad \text{and} \quad \mathbf{n}_C = -0.917\,\mathbf{N}_X - 0.398\,\mathbf{N}_Y \tag{76}$$

Then the cosine of the angle γ between the velocities of G and C is:

$$\cos\gamma = \mathbf{n}_G \bullet \mathbf{n}_C = -0.477 \tag{77}$$

VI. DISCUSSION

These results show that the assertion: "The cue ball goes in the direction of the cue stick" is valid if the ball is struck along the vertical great circle. If, however, the point of impact between the cue and the ball is slightly outside the vertical great circle plane, and if there is a downward orientation of the cue, the ball center will deviate from the horizontally projected cue direction and move on a parabolic path (Case 3). Although the deviation may appear to be small, it can have a significant effect upon the subsequent movement of the cue ball, and consequently, a deciding effect upon the movement of an object ball. In Parts II and III of this paper series we provide a quantification of these effects. Beyond this result and analysis, however, the major contribution of this paper is believed to be the solution algorithm of Section 4. With virtually no restrictions, this algorithm can accommodate all practical cue orientations and point of impact geometries. The output of the algorithm is a complete and accurate description of the post-cue impact kinematics of the cue ball. We will use these results as a starting point in our analysis of the post-cue-impact ball movement in Part II.

REFERENCES

[1] W.C. Marlow, *The physics of pocket billiards* (MAST [Marlow Advanced Systems Technologies], Palm Beach Gardens, FL, 1995).

[2] W. Mosconi, *On pocket billiards* (Crown Publishers, New York, NY, 1959).

[3] S. Mizerak, M.E. Panozzo, and G. Fels, *Play better pool* (Contemporary Books, Chicago, IL, 1996).

[4] B. Pejcic and R. Meyer, *Pocket billiards* (Sterling Publishing, New York, NY, 1993).

[5] D. G. Alciatore, *The illustrated principles of pool and billiards* (Sterling Publishing, New York, NY, 2004).

[6] T.R. Kane, *Dynamics* (Holt, Rinehard and Winston, New York, NY, 1968).

[7] R.L. Huston and C.Q. Liu, *Formulas for dynamic analysis* (Marcel Dekker, New York, NY, 2001).

[8] H. Josephs and R.L. Huston, *Dynamics of mechanical systems* (CRC Press, Boca Raton, FL, 2002).

[9] T.R. Kane, *Analytical elements of mechanics* (Academic Press, New York, NY, 1961) 211-216.