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A New Stiffness Matrix for Analysis Of Flexural Line Continuum

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Abstract

The analysis of line continuum by equilibrium approach to obtain exact results requires direct integration of the governing equations and satisfying the boundary conditions. This procedure is cumbersome and 4x4 stiffness matrix system with 4x1 load vector has been used as practical simplification for a long time. The problem with this system is that it is inefficient and does not give exact results, especially for classical studies that analyse only one element as a whole. This paper presents 5x5 stiffness matrix system as a new stiffness system to generate exact results. The stiffness matrix was developed using energy variational principle. The approach differs from the 4x4 stiffness matrix approach by considering a deformable node at the centre of the continuum which brings the number of deformable nodes to five. By carrying out direct integration of the governing differential equation of the line continuum, a five-term shape function was obtained. This was substituted into the strain energy equation and the resulting functional was minimized, resulting in a 5x5 stiffness matrix used herein. The five term shape function was also substituted into load-work equation and minimized to obtain load vector for uniformly distributed load on the continuum. Classical studies of beams of four different boundary conditions under uniformly distributed load and point load were performed using the new 5x5 stiffness matrix with 5x1 load vector for one element analysis. Numerical studies were also done for two elements. The process was repeated using the conventional 4x4 stiffness matrix and 4x1 load vector. It was found that unlike the conventional 4x4 stiffness system, the new 5x5 stiffness system is efficient for both classical and numerical analysis of beams as its results agree with exact results. It is therefore recommended that the new 5x5 stiffness system should be used for both classical and numerical analysis of line continuum.

***Key words**: 5x5 stiffness system; line continuum; variational principle; deformable node; shape function; load vector; classical; numerical; analysis; beam*

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I. INTRODUCTION

Element stiffness method of analyzing beams and other flexural line continua has been greatly used since Clough published a paper titled "The finite element method in plane stress analysis" (Clough, 1960). Many scholars subsequently investigated and embraced discrete element analysis as a means of finding approximate solutions to problems that are cumbersome to solve by the Euler or equilibrium approach of direct integration of the governing differential equation and simultaneously satisfying the boundary conditions (Melosh, 1963; Long, 1978; 1992; Cook et al., 1989; Huebner et al., 1995; Bathe, 1996; Zienkiewicz and Taylor, 2000).

For line continuum under flexural deformation – pure bending, buckling and vibration – the general governing differential equation is $\frac{d^4W(x)}{dx^4} + \frac{N}{EI}\frac{d^2W(x)}{dx^2} - \frac{q}{EI} + \frac{m\lambda^2}{EI}W(x) = 0$

For pure bending analysis, the governing equation reduces to $\frac{d^4W(x)}{dx^4} - \frac{q}{EI} = 0.$

For buckling, the equation becomes $\frac{d^4W(x)}{dx^4} + \frac{N}{EI}\frac{d^2W(x)}{dx^2} = 0.$ When free vibration problem is encountered, the governing equation becomes $\frac{d^4W(x)}{dx^4} + \frac{m\lambda^2}{EI}W(x) = 0$

The direct integration of these equations will result to exact general solution to the problems as the case may be. If the boundary conditions of a particular linear continuum are also satisfied in the general solution, then exact particular solution for the linear continuum will be obtained. Thus, obtaining exact particular solution for a linear continuum requires the direct integration of its governing differential equation and simultaneously satisfying its boundary conditions. Earlier scholars were used to trigonometric series formulation or approximation of the solution (deflection function or shape function). It was difficult to satisfy the boundary conditions using trigonometric series, so the exact approach was abandoned and energy and numerical approaches evolved (Iyengar, 1988). One of the numerical methods is the element stiffness method, which has gained wide acceptance in structural engineering. This method is based on minimum energy variation or weighted residual variation. In either case, it uses assumed shape function. The assumed shape function is chosen such that the number of terms in it must coincide with the number of chosen deformable nodes in the continuum (Reddy, 2005; Long et al., 2009). This practice is quite arbitrary as there is no limit to the number of terms to be used in the line continuum. Every scholar is free to choose as many terms as it suits his fancied number of deformable nodes. The whole idea is to come up with better approximating shape function, less rigorous computation and better results. In line with this objective, many shape functions and systems of deformable nodes line continuum were evolved. The most widely used is the four deformable-nodal system with Taylor-McLaurin's series shape function truncated at fourth term: $W(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$. As illustrated in figure 1, the four deformable nodes are (i) Deflection at left end of the continuum, (ii) rotation at left end of the continuum, (iii) deflection at right end of the continuum, and (iv) rotation at right end of the continuum. This system gave the stiffness matrix equation as:

$$\begin{bmatrix} QF1 \\ MF1 \\ QF2 \\ MF2 \end{bmatrix} = \begin{bmatrix} \frac{12EI}{L^3} & \frac{6EI}{L^2} & \frac{-12EI}{L^3} & \frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{4EI}{L} & \frac{6EI}{L^2} & \frac{2EI}{L} \\ \frac{-12EI}{L^3} & \frac{-6EI}{L^2} & \frac{12EI}{L^3} & \frac{-6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{2EI}{L} & \frac{-6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \begin{bmatrix} W_1 \\ \theta_1 \\ W_2 \\ \theta_2 \end{bmatrix}$$

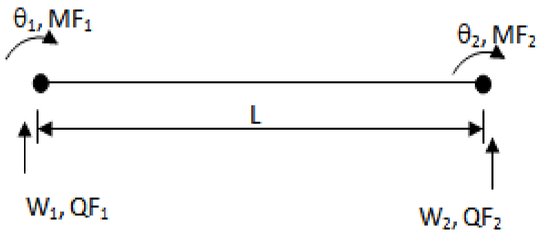


Figure 1: Four-deformable nodal of 4x4 stiffness system

This 4x4 matrix system has been widely accepted and used as the most suitable system for linear continuum analysis in spite of other advanced systems that have emerged, such as (i) four-degree of freedom quadratic spline beam element system and (ii) six-degree of freedom cubic spline beam element system (Long et al., 2009) as illustrated in figures 2 and 3. The acceptance of the 4x4 matrix system has to do with the ease at which the system is applied. A close look at the three systems described here (the 4x4 system and the two advanced systems) shows that only deformations at the ends of the linear continuum were considered. Notwithstanding these advances, it is still difficult to achieve good results for most linear continuum problems, especially in classical cases where many discrete elements are involved. It is this difficulty that drew the attention of this present paper.

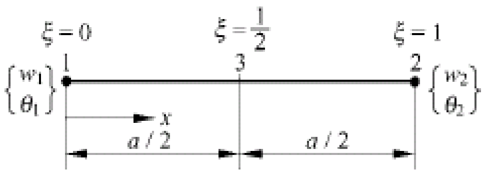


Figure 2

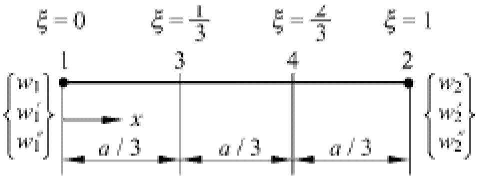


Figure 3

The governing differential equation for pure bending of line continuum is first integrated to obtain a general solution of the continuum that has specific number of terms. The general solution obtained is used in energy variational principle to get a new and more reliable 5x5 stiffness matrix for linear continuum. Results obtained from analysis of line continuum using the new 5x5 matrix and the conventional 4x4 matrix are compared with exact results.

II. DIRECT INTEGRATION OF GOVERNING EQUATION

The line continuum governing equation for pure bending is:

$$\frac{d^4 W(x)}{dx^4} - \frac{q}{EI} = 0 \qquad (1)$$

Integrating this equation four consecutive times with respect to x gave:

$$EIW(x) = \frac{qx^4}{24} + \frac{C_3X^3}{6} + \frac{C_2X^2}{2} + C_1X + C_0 \qquad (2)$$

Equation (2) can be written as:

$$W(x) = a_0 + a_1X + a_2X^2 + a_3X^3 + a_4X^4 \qquad (3)$$

$$Where\ a_0 = \frac{C_0}{EI};\ a_1 = \frac{C_1}{EI};\ a_2 = \frac{C_2}{2EI};\ a_3 = \frac{C_3}{6EI};\ a_4 = \frac{q}{24EI}$$

III. Energy Variational Principle

From Naschie (1990), strain energy is given as:

$$U_s = \frac{EI}{2}\int_0^L (W'')^2\, dx \qquad (4)$$

Work performed by load is the summation of all the works individually performed by each load on the continuum defined as the product of the individual load and the displacement it made during the course of bending of the line continuum. As example, work from uniformly distributed load, q on the continuum is:

$$U_{lu} = q\int_0^L W\, dx \qquad (5)$$

Work performed by n number of point loads, P_i at different positions on the continuum is:

$$U_{lpi} = \sum_{i=1}^{n} P_i . W_i \qquad (7)$$

Work performed by n number of moment loads, M_i at different positions on the continuum is:

$$U_{lpi} = \sum_{i=1}^{n} M_i . \theta_i \qquad (7)$$

Where W, W_i, and θ_i are shape function (deflection equation), deflection value at position i, and rotation (slope) value at position i respectively.

IV. New Flexural Stiffness Matrix For Line Continuum

We propose a new stiffness matrix system for line continuum which is different from the three highlighted earlier. It is based on the general solution of equation (3) that has five terms. Thus, the new matrix system is compelled to provide five deformable nodes. Two deformable nodes (deflection and rotation) each at the ends of the continuum give a total of four nodes. One more node is required to complete the system. Here, the fifth node (deflection) is provided at the mid span of the continuum. The resultant deformable nodes and corresponding forces as illustrated in figure 4 are:

$$[Nodes] = [w_1 \quad \theta_1 \quad w_2 \quad w_3 \quad \theta_3]$$

$$[FEA] = [QF_1 \quad MF_1 \quad QF_2 \quad MF_3 \quad QF_3]$$

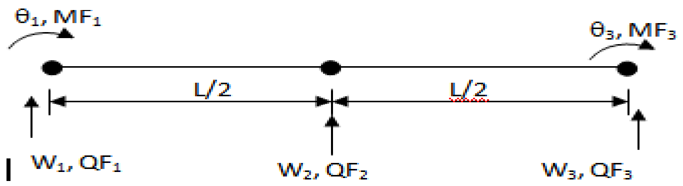


Figure 4: Five-deformable nodal system of 5x5 stiffness system

Substituting equation (3) into equation (4) and minimizing the resulting strain energy gives the new 5 x 5 stiffness matrix of line continuum as:

$$[K] = \begin{bmatrix} \frac{316EI}{5L^3} & \frac{94EI}{5L^2} & \frac{-512EI}{5L^3} & \frac{196EI}{5L^3} & \frac{-34EI}{5L^2} \\ \frac{94EI}{5L^2} & \frac{36EI}{5L} & \frac{-128EI}{5L^2} & \frac{34EI}{5L^2} & \frac{-6EI}{5L} \\ \frac{-512EI}{5L^3} & \frac{-128EI}{5L^2} & \frac{1024EI}{5L^3} & \frac{-512EI}{5L^3} & \frac{128EI}{5L^2} \\ \frac{196EI}{5L^3} & \frac{34EI}{5L^2} & \frac{-512EI}{5L^3} & \frac{316EI}{5L^3} & \frac{-94EI}{5L^2} \\ \frac{-34EI}{5L^2} & \frac{-6EI}{5L} & \frac{128EI}{5L^2} & \frac{-94EI}{5L^2} & \frac{36EI}{5L} \end{bmatrix} \qquad (8)$$

Similarly, substituting equation (3) into equation (5) and minimizing the resulting load work gives the 5 x 1 fixed end load vector [FEA] as:

$$[FEA] = \begin{bmatrix} QF_1 \\ MF_1 \\ QF_2 \\ QF_3 \\ MF_3 \end{bmatrix} = \begin{bmatrix} \frac{7qL}{30} \\ \frac{qL^2}{60} \\ \frac{40qL}{75} \\ \frac{7qL}{30} \\ \frac{-qL^2}{60} \end{bmatrix} \qquad (9)$$

V. Classical And Numerical Application

The results of the classical and numerical analysis of beams of four different boundary conditions using the new 5x5 stiffness matrix and the conventional 4x4 stiffness matrix are presented on tables 1 and 2 for uniformly distributed loaded beam and beam under central point load respectively. It can be seen from table 1 that although the conventional 4x4 system is efficient in numerical analysis (more than one element analysis – in the case herein, two elements) of the four types of beam under uniform load, it can scarcely analyze simply supported (P-R) beam, propped cantilever (C-R) beam, and cantilever (C-F) beam by classical (one element analysis or analytical) means. It can not be used for classical analysis of clamped (C-C) beam. On the other hand, the new 5x5 stiffness system is efficient for analysing the four types of beam under uniform load. It can be used either for classical analysis or numerical analysis as results from the system compare very well with exact results. Also, table 2 shows that for beam under central point load the conventional 4x4 stiffness system is efficient when used numerically but can only scarcely analyse cantilever beam classically. On the other hand, the new 5x5 stiffness system is efficient in both classical and numerical analysis of beams under central point load as it gives results that compare very well with the exact results.

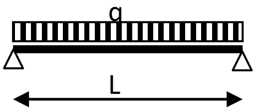
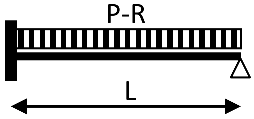
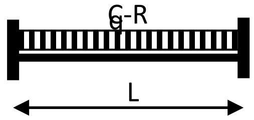
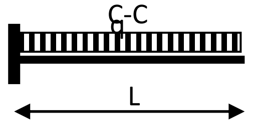
VI. CONCLUSSIONS

We conclude that the conventional 4x4 stiffness matrix system is suitable for numerical analysis of beams but unsuitable for classical analysis. On the other hand, the new 5x5 stiffness system is suitable for both numerical and classical analyses of beams.

VII. Recommendations

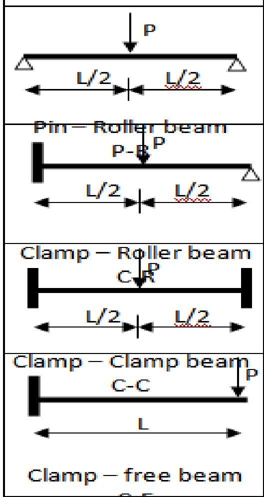
- **i.** The new 5x5 stiffness matrix system should be used for classical analysis of beams since classical analysis is easier and less time consuming than numerical analysis.
- ii. Future research should try 5x5 stiffness system for classical study of column (strut) stability.
- iii. Future research should consider 5x5 stiffness system for classical vibration analysis of beams.

Table 1: displacements of beam under uniformly distributed load

LOADED BEAM	Dis-place-ment	EXACT Result	One Element analysis (Classical) Result		Two Element analysis (Numerical) Result	
			4X4 STIFFNESS	5X5 STIFFNESS	4X4 STIFFNESS	5X5 STIFFNESS
 Pin – Roller beam	YL		0	0		
	YC	$\frac{-5}{384}$	NA	$\frac{-5}{384}$	$\frac{-5}{384}$	$\frac{-5}{384}$
	YR	0	0	0	0	0
	θL	$\frac{-1}{24}$	$\frac{-1}{24}$	$\frac{-1}{24}$	$\frac{-1}{24}$	$\frac{-1}{24}$
	θC	0	NA	NA	0	0
	θR	$\frac{1}{24}$	$\frac{1}{24}$	$\frac{1}{24}$	$\frac{1}{24}$	$\frac{1}{24}$
P-R  Clamp – Roller beam	YL	0	0	0	0	0
	YC	$\frac{-1}{192}$	NA	$\frac{-1}{192}$	$\frac{-1}{192}$	$\frac{-1}{192}$
	YR	0	0	0	0	0
	θL	0	0	0	0	0
	θC	$\frac{-1}{192}$	NA	NA	$\frac{-1}{192}$	$\frac{-1}{192}$
	θR	$\frac{1}{48}$	$\frac{1}{48}$	$\frac{1}{48}$	$\frac{1}{48}$	$\frac{1}{48}$
C-R  Clamp – Clamp beam	YL	0	NA	0	0	0
	YC	$\frac{-1}{384}$	NA	$\frac{-1}{384}$	$\frac{-1}{384}$	$\frac{-1}{384}$
	YR	0	NA	0	0	0
	θL	0	NA	0	0	0
	θC	0	NA	NA	0	0
	θR	0	NA	0	0	0
C-C  Clamp – free beam C-F	YL	0	0	0	0	0
	YC	$\frac{-51}{1152}$	NA	$\frac{-51}{1152}$	$\frac{-51}{1152}$	$\frac{-51}{1152}$
	YR	$\frac{-1}{8}$	$\frac{-1}{8}$	$\frac{-1}{8}$	$\frac{-1}{8}$	$\frac{-1}{8}$
	θL	0	0	0	0	0
	θC	$\frac{-21}{144}$	NA	NA	$\frac{-21}{144}$	$\frac{-21}{144}$
	θR	$\frac{-1}{6}$	$\frac{-1}{6}$	$\frac{-1}{6}$	$\frac{-1}{6}$	$\frac{-1}{6}$

Legend:
YL, YC and YR represent deflections at left end, centre and right end of the beam respectively.
θL, θC and θR represent rotations at left end, centre and right end of the beam respectively.
Unit of measurement for deflection and rotation are $\frac{qL^4}{EI}$ and $\frac{qL^4}{EI}$ respectively.
NA mean not accessible from the system

Table 2: displacements of beam under point load



LOADED BEAM	Dis-place-ment	EXACT Result	One Element analysis (Classical) Result		Two Element analysis (Numerical) Result	
			4X4 STIFFNESS	5X5 STIFFNESS	4X4 STIFFNESS	5X5 STIFFNESS
Pin – Roller beam P-R	YL	0	NA	0	0	0
	YC	$\frac{-1}{48}$	NA	-0.020508	$\frac{-1}{48}$	$\frac{-1}{48}$
	YR	0	NA	0	0	0
	θL	$\frac{-1}{16}$	NA	$\frac{-1}{16}$	$\frac{-1}{16}$	$\frac{-1}{16}$
	θC	0	NA	NA	0	0
	θR	$\frac{1}{16}$	NA	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$
Clamp – Roller beam C-R	YL	0	NA	0	0	0
	YC	$\frac{-7}{768}$	NA	-0.00879063	$\frac{-7}{768}$	$\frac{-7}{768}$
	YR	0	NA	0	0	0
	θL	0	NA	0	0	0
	θC	$\frac{-1}{128}$	NA	NA	$\frac{-1}{128}$	$\frac{-1}{128}$
	θR	$\frac{1}{32}$	NA	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{32}$
Clamp – Clamp beam C-C	YL	0	NA	0	0	0
	YC	$\frac{-1}{192}$	NA	$\frac{-5}{1024}$	$\frac{-1}{192}$	$\frac{-1}{192}$
	YR	0	NA	0	0	0
	θL	0	NA	0	0	0
	θC	0	NA	NA	0	0
	θR	0	NA	0	0	0
Clamp – free beam C-F	YL	0	0	0	0	0
	YC	$\frac{-5}{48}$	NA	$\frac{-5}{48}$	$\frac{-5}{48}$	$\frac{-5}{48}$
	YR	$\frac{-1}{3}$	$\frac{-1}{3}$	$\frac{-1}{3}$	$\frac{-1}{3}$	$\frac{-1}{3}$
	θL	0	0	0	0	0
	θC	$\frac{-9}{24}$	NA	NA	$\frac{-9}{24}$	$\frac{-9}{24}$
	θR	$\frac{-1}{2}$	$\frac{-1}{2}$	$\frac{-1}{2}$	$\frac{-1}{2}$	$\frac{-1}{2}$

Legend:
YL, YC and YR represent deflections at left end, centre and right end of the beam respectively.
θL, θC and θR represent rotations at left end, centre and right end of the beam respectively.
Unit of measurement for deflection and rotations are $\frac{PL^3}{EI}$ and $\frac{PL^2}{EI}$ respectively.
NA mean not accessible from the system

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