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The Stability of the Solution of Nonlinear Integral Equation in Two-Dimensional Problems

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Abstracts: *Here, the existence and uniqueness of the solution of nonlinear **F-VIE** is considered, then we use a numerical method to reduce this type of equation to a system of **FIE**, Trapezoidal method and Simpson's method are applied to solve the **FIE** of the second kind with continuous kernel. Error in each case is calculated to elucidate the accuracy of this methods.*
***Key Words**: Two-Dimensional Problems- Nonlinear Fredholm -Volterra integral equation- Trapezoidal and Simpson's methods- polynomials- Continuous kernels.*

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I. INTRODUCTION

The mixed integral equation of Fredholm -Volterra type can be solved analytically using these methods: Cauchy method, orthogonal polynomial method, Potential theory method and Krein's method. The importance of **F-VIE** of the first kind and contact problems came from Abdou work in [1]. Where, the solution of **F-VIE** of the first kind in one, two and three dimensional has been obtained, analytically using separation of variables method.

Now, the solution of **F– VIE** of the second kind plays an important rule in contact problem and theory of elasticity. In [2], a numerical method is used to obtain the solution of **F– VIE** of the second kind when the kernel of Fredholm integral term takes a form of continuous function. Also the relations between the **F–VIE** and contact problems have been discussed in [3]. Using the Fredholm index and operator theory, the existence and uniqueness of **F-VIE** of the second kind have been obtained in [4].

In this paper we will discuss and solve the nonlinear **F-VIE** of the second kind using two different methods. **FI** term is considered in position, while **VI** term is measured in time.

The remainder of this paper is devoted into three sections. In section two the stability of the solution of nonlinear **F-VIE** of the second kind in the space $L_2[0,1]\times C[0,T],T<1$, are presented. The kernel of **FI** term is considered as a continuous function in position, while the kernel of **VI** term is a positive continuous function in time. In section three, using a numerical method, the **F-VIE** will be reduced to a linear algebraic system of **FIE**. In section four Trapezoidal and Simpson's methods are discussed and applied to solve the **FIE** of the second kind with continuous kernels. Finally, conclusion is considered to explain the results.

II. THE STABILITY OF THE SOLUTIONS

Consider the formula,

$$\mu\varphi(x,t)=\gamma(f(x,t),\varphi(x,t))+\lambda\int_0^1 k(x,y)\,\varphi(y,t)\,dy+\lambda\int_0^t F(t,\tau)\,\varphi(x,\tau)\,d\tau \qquad (2.1)$$

The integral equation (2.1) is called nonlinear **F-V** integral type. This formula is considered in the space $L_2[0,1]\times C[0,T],T<1$, where the Fredholm integral term is considered in position and its kernel $k(x,y)$ may has a singular term. While the Volterra integral term is considered in time and its kernel $F(t,\tau)$ is positive and continuous for all $t,\tau\in[0,T],T<1$, μ is defined the kind of integral equation and λ is a constant that has a physical meaning.

In order to guarantee the existence of a unique solution of equation (2.1) by using Picard method, we assume the following conditions:

(i) The kernel of the Fredholm term $k(x,y)\in C\left([0,1]\times[0,1]\right)$, and for the discontinuous case, we use the Fedholm condition:

$$\left[\int_0^1\int_0^1 |k(x,y)|^2\,dy\,dx\right]^{\frac{1}{2}}=c,\quad (c \text{ is a constant })$$

(ii) The kernel of the Volterra term $F(t,\tau)\in C\left([0,T]\times[0,T]\right),\ 0\le\tau\le t\le T<1$, and satisfies: $\left|F(t,\tau)\right|\le M,\ \forall\, t,\tau\in[0,T],\quad M$ is a constant

(iii) The given function $\gamma(f(x,t),\varphi(x,t))$ with its partial derivatives with respect to x and t are continuous in $L_2[0,1]\times C[0,T]$, and satisfies,

(iii-a) $\quad |\gamma(x,t,f,\varphi)|\le \ell+\alpha|\varphi|$

(iii-b) $\quad \|\gamma(f,\varphi)\|=\max\limits_{0\le t\le T}\int_0^t\left[\int_0^1|\gamma(f,\varphi)|^2\,dx\right]^{\frac{1}{2}}d\tau=H,\ H$ is a constant

(iii-c) $\quad |\gamma(f,\varphi_1)-\gamma(f,\varphi_2)|\le A|\varphi_1-\varphi_2|.(A<1)$

Where

$$\varphi_1\ ,\ \varphi_2\ \in L_2[0,1]\times C[0,T]$$

To prove that the solution is exist, we use the successive approximation, for this we pick up any real continuous function $\phi_0(x,t)$ in $L_2[0,1]\times C[0,T]$, then construct a sequence ϕ_n defined by

$$\mu\varphi_n(x,t)=\gamma(f(x,t),\varphi_n(x,t))+\lambda\int_0^1 k(x,y)\,\varphi_{n-1}(y,t)\,dy+\lambda\int_0^t F(t,\tau)\,\varphi_{n-1}(x,\tau)\,d\tau$$

$$\mu\varphi_{n-1}(x,t)=\gamma(f(x,t),\varphi_{n-1}(x,t))+\lambda\int_0^1 k(x,y)\,\varphi_{n-2}(y,t)\,dy+\lambda\int_0^t F(t,\tau)\,\varphi_{n-2}(x,\tau)\,d\tau \qquad (2.2)$$

For ease of manipulation it is convenient to introduce:

$$\psi_n(x,t)=[\gamma(f(x,t),\varphi_n(x,t))-\gamma(f(x,t),\varphi_{n-1}(x,t))]$$

$$+\lambda\int_0^1 k(x,y)\left[\varphi_{n-1}(y,t)-\varphi_{n-2}(y,t)\right]dy+\lambda\int_0^t F(t,\tau)\left[\varphi_{n-1}(x,\tau)-\varphi_{n-2}(x,\tau)\right]d\tau,\quad n=1,2 \qquad (2.3)$$

Where, $\qquad \psi_n(x,t)\le\left(\varphi_n-\varphi_{n-1}\right) \qquad (2.4)$

the formula (2.4) can be adapted in the form,

$$\varphi_n(x,t)=\sum_{i=1}^{n}\psi_i(x,t) \qquad (2.5)$$

Using the conditions (iii-c) and (1.3) we have,

$$\beta_1\psi_n(x,t)=\lambda\int_0^1 k(x,y)\,\psi_{n-1}(y,t)\,dy+\lambda\int_0^t F(t,\tau)\,\psi_{n-1}(x,\tau)\,d\tau,\qquad (\beta_1=1-A) \qquad (2.6)$$

Using the properties of the norm and conditions, (i) and (iii) to get,

$$\beta_1\|\psi_1(x,t)\|\le|\lambda|\max_{0\le t\le T}\int_0^t\left[\int_0^1\left(\int_0^1|k(x,y)|^2\,dy\int_0^1|\psi_0(y,t)|^2\,dy\right)dx\right]^{\frac{1}{2}}d\tau+|\lambda|M\int_0^t\|\psi_0(x,\tau)\|\,d\tau$$

$$\le|\lambda|c\max_{0\le t\le T}\int_0^t\left[\int_0^1|\psi_0(y,t)|^2\,dy\right]^{\frac{1}{2}}d\tau+|\lambda|\,M\,\ell\|\,t\,\| \tag{2.7}$$

Finally, we get,

$$\|\psi_n(x,t)\|\le|\lambda|^n\;\ell\left(\frac{c+M\,T}{\beta_1}\right)^n=\ell\,\alpha^n,\qquad \alpha=|\lambda|\left(\frac{c+M\,T}{\beta_1}\right) \tag{2.8}$$

This bound makes the sequence $\psi_n(x,t)$ converges under the following condition,

$$\alpha<1\quad\to\quad|\lambda|<\frac{\beta_1}{c+M\,T} \tag{2.9}$$

The result of (2.9), leads us to say that the formula (2.7) has a convergent solution. So, let $n\longrightarrow\infty$, we have

$$\varphi(x,t)=\sum_{i=0}^{\infty}\psi_i(x,t) \tag{2.10}$$

The infinite series of (2.10) is convergent, and $\phi(x,t)$ represents the convergent solution of Eq. (2.1). Also each of ψ_i is continuous, therefore $\phi(x,t)$ is also continuous.

To show that $\phi(x,t)$ is unique, assume that $\tilde{\phi}(x,t)$ is also a continuous solution of (4.1) then, we write

$$\varphi(x,t)-\tilde{\varphi}(x,t)=[\gamma(f(x,t),\varphi(x,t))-\gamma(f(x,t),\tilde{\varphi}(x,t))] \tag{2.11}$$

$$+\lambda\int_0^1 k(x,y)[\varphi(y,t)-\tilde{\varphi}(y,t)]dy+\lambda\int_0^t F(t,\tau)\,[\varphi(x,\tau)-\tilde{\varphi}(x,\tau)]d\tau$$

This leads us to the following

Using condition (i) and (ii) , then applying Cauchy - Schwarz inequality, equation (2.11) describe the conditions for stability solution.

We obtain,

$$\beta_1\|\varphi(x,t)-\tilde{\varphi}(x,t)\|\le|\lambda|c\|\varphi(y,t)-\tilde{\varphi}(y,t)\|+|\lambda|M\,T\,\|\varphi(x,\tau)-\tilde{\varphi}(x,\tau)\| \tag{2.12}$$

$$\le\alpha\|\varphi(y,t)-\tilde{\varphi}(y,t)\|$$

Finally, we obtain

$$(1-\alpha)\|\varphi(x,t)-\tilde{\varphi}(x,t)\|\le 0,\qquad \alpha<1 \tag{2.13}$$

Since $\|\phi(x,t)-\tilde{\phi}(x,t)\|$ is necessarily non – negative, and $\alpha<1$, we get

$$\|\varphi-\tilde{\varphi}\|=0\quad\Rightarrow\quad\varphi=\tilde{\varphi}.$$

It follows that if (2.1) has a unique solution.

III. NUMERICAL METHODS AND EXAMPLES,

In this section, we discuss the solution of the nonlinear **NF-VIE** (1.1) numerically using two different methods Trapezoidal rule, Simpson rule method, and determine the error in each method.

● **Trapezoidal rule:** For solving equation (1.1) numerically**,** we divide the interval $[0,1]$ into s subintervals with length $h=1/s$; $s=0,1,2,3,...L$ can be even or odd, where $x=x_i, y=x_j, \quad 0<i,j<s$.
Then the nonlinear **F-VIE** (1.1) reduce to the following nonlinear algebraic system
When u_j the weight function with respect to position, while w_ζ weight function with respect to time.

$$\varphi_n(x_i)=\gamma_n(f(x_i),\varphi(x_i))+\lambda\sum_{j=0}^{s}u_j k(x_i,x_j)\varphi(x_j) \tag{3.1}$$

$$+\lambda\sum_{\zeta=0}^{n}w_\zeta F_{n,\zeta}\varphi_\zeta(x_i)+R_{n,s}, \quad n=0,1,2,\cdots,N.$$

Where, u_j, w_ζ are the weights defined by

$$u_j=\begin{cases} h/2 & j=0,S \\ h & 0<j<S. \end{cases}; \qquad w_\zeta=\begin{cases} h/2 & \zeta=0,n \\ \\ h & 0<\zeta<n \end{cases}$$

After neglecting the error, we using the following notations:
$\varphi_{n,i}=\varphi_n(x_i),\ \gamma_{n,i}(f_{n,i},\varphi_{n,i})=\gamma_n(f_n(x_i),\varphi_n(x_i)),\ k_{i,j}=k(x_i,x_j), F_{\ell,m}=F(t_\ell,t_m)$. The formula (1.1) can be rewritten in the following form:

$$\varphi_{n,i}=\gamma_{n,i}(f_{n,i},\varphi_{n,i})+\lambda\sum_{j=0}^{s}u_j k_{i,j}\varphi_{n,j}+\lambda\sum_{\zeta=0}^{n}w_\zeta F_{n,\zeta}\varphi_{\zeta,i}+R_{n,S} \quad ,0\le n\le S. \tag{3.2}$$

The formula (2.1) represents system of $(N+1)$ equations and $(N+1)$ unknowns coefficients. By solving them, we can obtain the approximation solution of (1.1).

Definition 1: The estimate the error R_N of **Trapezoidal rule** is determined by

$$R_{n,s}=\left|\begin{array}{l}\gamma(f(x,t),\varphi(x,t))+\lambda\int_0^1 k(x,y)\varphi(y;t)\,dy+\lambda\int_0^t F(t,\tau)\varphi(x;\tau)\,d\tau \\ -\gamma_{n,i}(f_{n,i},\varphi_{n,i})-\lambda\sum_{j=0}^{s}u_j k_{i,j}\phi_{n,j}-\lambda\sum_{\zeta=0}^{n}w_\zeta F_{n,\zeta}\phi_{i,\zeta}\end{array}\right|, \quad 0\le i\le s \tag{3.3}$$

Theorem (without proof): the nonlinear system (2.1) has a unique solution in the space ℓ_∞ under the conditions:

$$|\lambda|<\frac{\beta_1'}{L'+M'};\ \beta_1'=1-\ell_2' \tag{3.4}$$

Where, for a constant $\ell'=\max\{\ell_1',\ell_2',\ell_3'\}$, we have consider the definition of,

$$\left.\begin{array}{l}(1')\left|\gamma_{n,i}(f_{n,i},\varphi_{n,i})\right|\le\ell_1'+\ell_2'\left|\varphi_{n,i}\right| \\ (2')\left|\gamma_{n,i}(f_{n,i},\varphi_{n,i})-\right|\left|\gamma_{n,i}(f_{n,i},\psi_{n,i})\right|\le\ell_3'\left|\varphi_{n,i}-\psi_{n,i}\right|\end{array}\right\}$$

Also, we have

$$(ii'')\sup\sum_{j=0}^{S}\left|\rho_j k_{i,j}\phi_{n,j}\right|\le L'',\ (iii'')\sup\sum_{\zeta=0}^{n}\left|\vartheta_\varsigma F_{n,\zeta}\phi_{i,\varsigma}\right|\le M' \tag{3.5}$$

Simpson rule:

For using Simpson rule to solve (2.1) numerically, we divide the interval $[0,1]$ into s subintervals with length $h=1/s$ is even, $0\le i,j\le s.$ Let $x=x_i, y=x_j$, then, after approximating the integrals term, we get

$$\varphi_{n,i} = \gamma_{n,i}(f_{n,i}, \varphi_{n,i}) + \lambda \sum_{j=0}^{s} \rho_j k_{i,j} \varphi_j + \lambda \sum_{\zeta=0}^{n} \vartheta_\zeta F_{n,\zeta} \varphi_{\zeta,i}, \quad 0 \le n \le s \tag{3.6}$$

$$R_N = \left| \int_0^1 k(x,y)\varphi(y,t)dy - \lambda \sum_{j=0}^{s} \rho_j k_{i,j} \varphi_{\varsigma,j} \right|$$

Where the weight ρ_j is defined as

$(\rho_j = h/3, j = 0,s);(\rho_j = 4h/3, 0 < j < s, j\ odd)$, and $(\rho_j = 2h/3, 0 < j < s,\ j\ even)$

While, the weight ϑ_ζ takes two forms depending on the value of n odd or even

If n is odd we use Trapezoidal rul**e** and then,

$$\vartheta_\zeta = \omega_\zeta;\ (\omega_\zeta = h/2,\ \zeta = 0,n);(\omega_\zeta = h,\ 0 < \zeta < n)\ and\ (\omega_\zeta = 0, \zeta > n) \tag{3.7}$$

(2.8)

If n is even we use Simpson rul**e** and

$$\vartheta_\zeta = \omega_\zeta;\ (\omega_\zeta = h/3,\ \zeta = 0,n);(\omega_\zeta = 4h/3,\ 0 < \zeta < n, \zeta\ odd); \tag{3.8}$$

$$(\omega_\zeta = 2h/3,\ 0 < \zeta < n, \zeta\ even)\ and\ (\omega_\zeta = 0, \zeta > n)$$

Definition 2: The estimate error of **Simpson rule** is determined by

$$\left| R_{n,s} \right| = \left| \gamma(f(x,t), \varphi(x,t)) + \lambda \int_0^1 k(x,y)\varphi(y,t)\,dy + \lambda \int_0^t F(t,\tau)\varphi(x,\tau)\,d\tau \right. \tag{3.9}$$

$$\left. - \{\gamma_{n,i}(f_{n,i}, \varphi_{n,i}) + \lambda \sum_{j=0}^{s} \rho_j k_{i,j} \varphi_j + \lambda \sum_{\zeta=0}^{n} \vartheta_\zeta F_{n,\zeta} \varphi_{\zeta,i}\} \right|$$

Theorem (without proof): The nonlinear system (2.1) using Trapezoidal and Simpson's methods has a unique solution in space ℓ_∞, under the conditions, (2.6)

$$|\lambda| < \frac{B'}{L'' + M''};\ B' = 1 - \ell_2'$$

If $n, s \to \infty$, we have $R_{n,s} \to 0$

$$\left\{ \gamma_{n,i}(f_{n,i}, \varphi_{n,i}) + \lambda \sum_{j=0}^{S} u_j k_{i,j} \phi_{n,j} + \lambda \sum_{\varsigma=0}^{n} w_\varsigma F_{n,\varsigma} \phi_{i,\varsigma} \right\} \to$$

$$\left\{ \gamma(f(x,t), \varphi(x,t)) + \lambda \int_0^1 k(x,y)\varphi(y;t)\,dy + \lambda \int_0^t F(t,\tau)\varphi(x;\tau)\,d\tau \right\}$$

IV. NUMERICAL EXAMPLES:

Example (1)

$$\mu\varphi(\mathrm{x,t}) = \mathrm{f(x,t)} + \beta\varphi^2(\mathrm{x,t}) - \lambda \int_0^1 x^2 y \varphi(\mathrm{y,t})\mathrm{dy} - \lambda t \int_0^t t^2 \tau . \varphi(\mathrm{x}, \tau)\mathrm{d}\tau$$

$$(\varphi(\mathrm{x,t}) = x^2 t^2;\ t = 0.01, \mu = 0.8)$$

x	Exact Solution	Numerical Solution of Trapezoidal	Error of Trapezoidal	Numerical solution Of Simpson's	Error of Simpson's
0	0	0	0	0	0
0.05	3.906249×10^{-7}	3.906297×10^{-7}	4.783381×10^{-12}	3.906250×10^{-7}	3.398719×10^{-19}
0.1	0.00000156249	0.000001562519	1.913355×10^{-11}	0.000001562500	1.35991×10^{-18}
0.15	0.00000351562	0.000003515668	4.305060×10^{-11}	0.000003515625	3.059483×10^{-18}
0.2	0.00000624999	0.000006250076	7.653467×10^{-11}	0.000006250000	5.439645×10^{-18}
0.25	0.00000976562	0.000009765744	1.195859×10^{-10}	0.000009765625	8.495740×10^{-18}
0.3	0.00001406249	0.000014062672	1.722046×10^{-10}	0.000014062500	1.223793×10^{-17}
0.35	0.00001914062	0.000019140859	2.343912×10^{-10}	0.000019140625	1.665266×10^{-17}
0.4	0.00002499999	0.000025000306	3.061458×10^{-10}	0.000025000000	2.175519×10^{-17}
0.45	0.00003164062	0.000031641012	3.874690×10^{-10}	0.000031640625	2.752518×10^{-17}
0.5	0.00003906249	0.000039062978	4.783613×10^{-10}	0.000039062500	3.398973×10^{-17}
0.55	0.00004726562	0.000047266203	5.788231×10^{-10}	0.000047265625	4.113869×10^{-17}
0.6	0.00005624999	0.000056250688	6.888551×10^{-10}	0.000056250000	4.895172×10^{-17}
0.65	0.00006601562	0.000066016433	8.084578×10^{-10}	0.000066015625	5.744916×10^{-17}
0.7	0.00007656249	0.000076563437	9.376321×10^{-10}	0.000076562500	6.661067×10^{-17}
0.75	0.00008789062	0.000087891701	1.0763786×10^{-9}	0.000087890625	7.646335×10^{-17}
0.8	0.00009999999	0.000100001224	1.2246982×10^{-9}	0.000100000000	8.704788×10^{-17}

Table (1)

Table (1) describes the deference in errors using Trapezoidal and Simpson's rule at time t = 0.01.

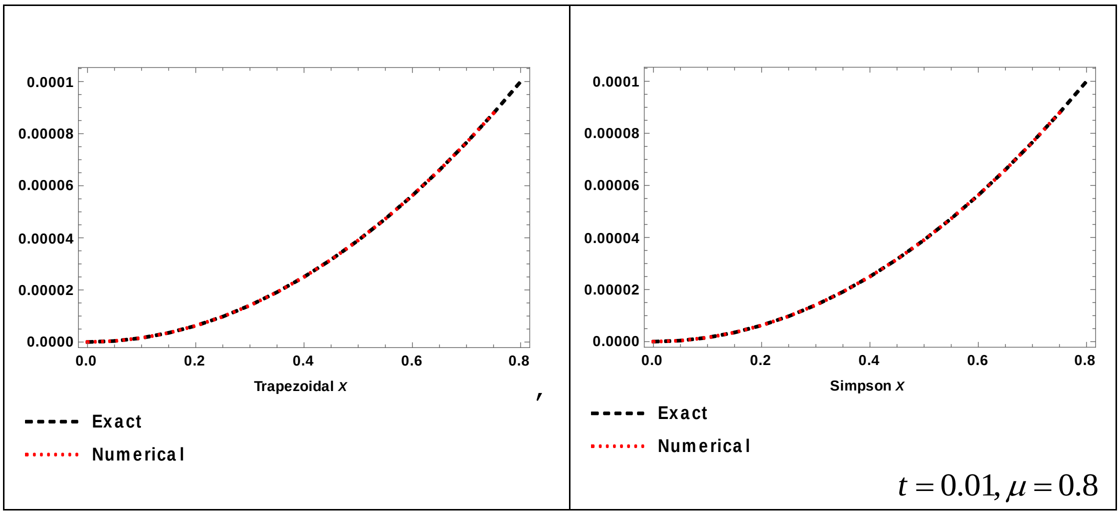

Figure (1)

Figure (1) describe the relationship between exact solution and numerical solution using Trapezoidal and Simpson's rule.

x	Exact Solution	Numerical Solution of Trapezoidal	Error of Trapezoidal	Numerical Solution of Simpson's	Error of Simpson's
0	0	0	0	0	0
0.05	0.0000390625	0.0000390629	4.787142×10^{-10}	0.0000390625	3.401728×10^{-13}
0.1	0.0001562500	0.0001562519	1.915137×10^{-9}	0.0001562500	1.360890×10^{-12}
0.15	0.0003515625	0.0003515668	4.310111×10^{-9}	0.0003515625	3.062751×10^{-12}
0.2	0.0006250000	0.0006250076	7.665042×10^{-9}	0.0006250000	5.446754×10^{-12}
0.25	0.0009765625	0.0009765744	1.198189×10^{-8}	0.0009765625	8.514298×10^{-12},
0.3	0.0014062500	0.0014062672	1.726321×10^{-8}	0.0014062500	1.226718×10^{-11}
0.35	0.0019140625	0.0019140860	2.351210×10^{-8}	0.0019140625	1.670762×10^{-11}
0.4	0.0025000000	0.0025000307	3.073225×10^{-8}	0.0025000000	2.183824×10^{-11}
0.45	0.0031640625	0.0031641014	3.892792×10^{-8}	0.0031640625	2.766205×10^{-11}
0.5	0.0039062500	0.0039062981	4.810396×10^{-8}	0.0039062500	3.418252×10^{-11}
0.55	0.0047265625	0.0047266207	5.826584×10^{-8}	0.0047265625	4.140352×10^{-11}
0.6	0.0056250000	0.0056250694	6.941959×10^{-8}	0.0056250000	4.932933×10^{-11}
0.65	0.0066015625	0.0066016440	8.157188×10^{-8}	0.0066015625	5.796472×10^{-11}
0.7	0.0076562500	0.0076563447	9.473000×10^{-8}	0.0076562500	6.731484×10^{-11}
0.75	0.0087890625	0.0087891714	1.089018×10^{-7}	0.0087890625	7.738532×10^{-11}
0.8	0.0100000000	0.0100001240	1.240960×10^{-7}	0.0100000000	8.818227×10^{-11}

Table (2)

Table (2) describes the deference in errors using Trapezoidal and Simpson's rule at fixed time t = 0.1

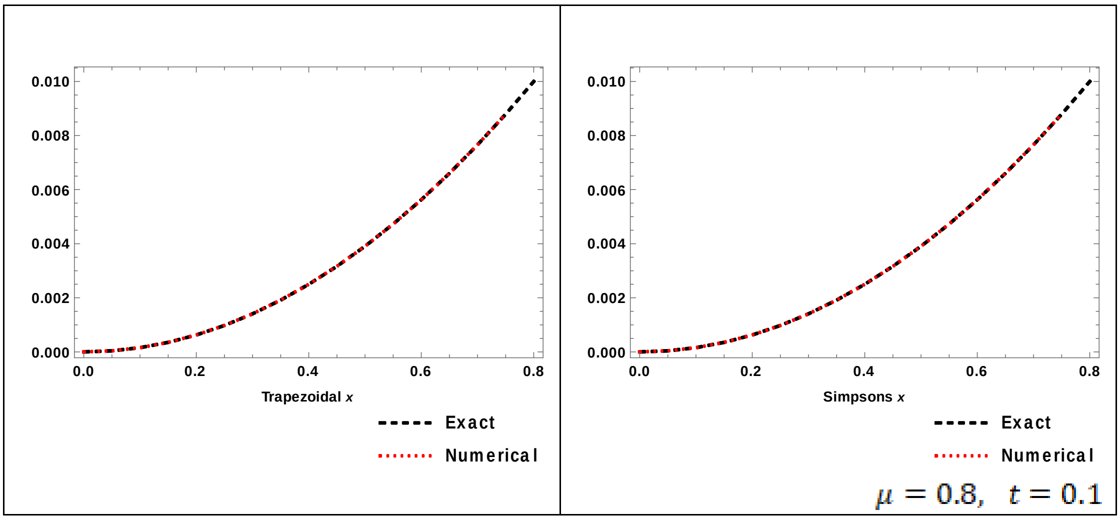

Figure 2

Figure (2) describe the relationship between exact solution and numerical solution using Trapezoidal and Simpson's rule.

x	Exact Solution	Numerical Solution of Trapezoidal	Error of Trapezoidal	Numerical Solution of Simpson's	Error of Simpson's
0	0	0	0	0	0
0.05	0.003525390	0.0035253660	2.456349×10^{-8}	0.003525866	4.763531×10^{-7}
0.1	0.014101562	0.0141014629	9.956699×10^{-8}	0.014103452	0.00000189012
0.15	0.031728515	0.0317282864	2.291295×10^{-7}	0.031732699	0.00000418425
0.2	0.056406250	0.0564058292	4.207641×10^{-7}	0.056413473	0.00000722388
0.25	0.088134765	0.0881340790	6.865367×10^{-7}	0.088145492	0.00001072732
0.3	0.126914062	0.1269130173	0.00000104515	0.126928208	0.00001414600
0.35	0.172744140	0.1727426148	0.00000152575	0.172760596	0.00001645586
0.4	0.225625000	0.2256228250	0.00000217492	0.225640779	0.00001577905
0.45	0.285556640	0.2855535697	0.00000307089	0.285565287	0.00000864666
0.5	0.352539062	0.3525347080	0.00000435445	0.352527476	0.00001158639
0.55	0.426572265	0.4265659600	0.00000630560	0.426513639	0.00005862648
0.6	0.507656250	0.5076466794	0.00000957052	0.507491581	0.00016466802
0.65	0.595791015	0.5957749663	0.00001604924	0.595366148	0.00042486726
0.7	0.690976562	0.6909417496	0.00003481280	0.689686874	0.00128968751
0.75	0.793212890	0.7925453367	0.00066755384	0.777778798	0.01543409256
0.8	0.902500000	0.6960957152	0.20640428471	0.693195590	0.20930440932

Table (3)

Table (3) describes the deference in errors using Trapezoidal and Simpson's rule at fixed time t = 0.95

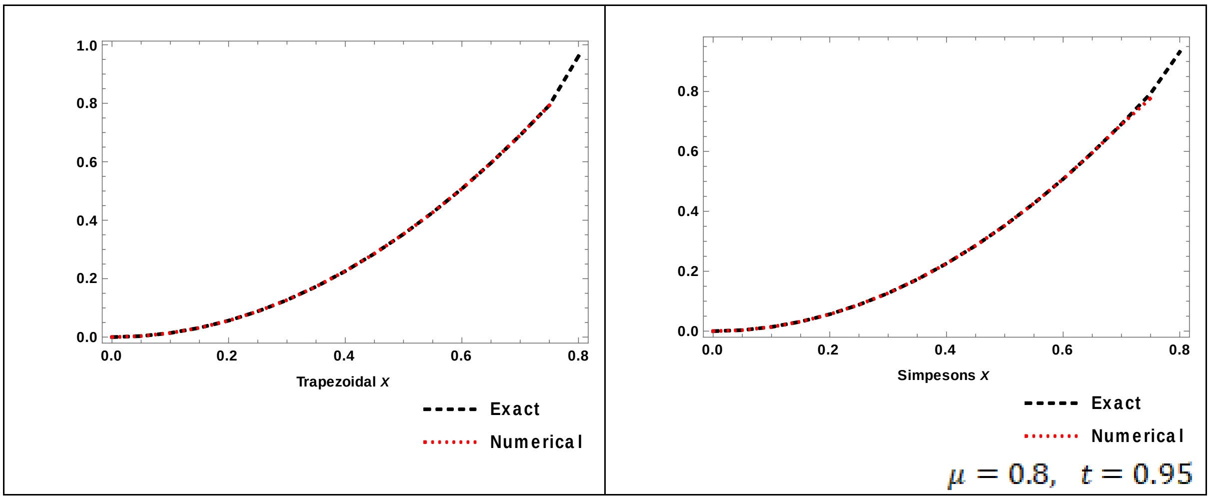

Figure (3)

Figure (3) describe the relationship between exact solution and numerical solution using Trapezoidal and Simpson's rule.

Example (2)

$$\mu\varphi(\mathrm{x},\mathrm{t}) = \mathrm{f}(\mathrm{x},\mathrm{t}) + \beta\varphi^2(\mathrm{x},\mathrm{t}) - \lambda\int_0^1 e^{xy}\varphi(\mathrm{y},\mathrm{t})\mathrm{dy} - \lambda t\int_0^t t^2\tau.\varphi(\mathrm{x},\tau)\mathrm{d}\tau$$

$$(\varphi(\mathrm{x},\mathrm{t}) = x^2t^2;\ t = 0.01, \mu = 0.8)$$

x	Exact Solution	Numerical Solution of Trapezoidal	Error of Trapezoidal	Numerical Solution of Simpson's	Error of Simpson's
0	0	0	0	0	0
0.05	3.9062×10^{-7}	3.91542×10^{-7}	9.1723×10^{-10}	3.9062×10^{-7}	1.6389×10^{-14}
0.1	0.0000015624	0.00000156350	1.0046×10^{-9}	0.0000015625	3.3786×10^{-14},
0.15	0.0000035156	0.00000351672	1.0994×10^{-9},	0.0000035156	6.4689×10^{-14},
0.2	0.0000062499	0.00000625120	1.2023×10^{-9}	0.0000062500	1.1121×10^{-13}
0.25	0.0000097656	0.00000976693	1.3140×10^{-9},	0.0000097656	1.7573×10^{-13}
0.3	0.0000140624	0.00001406393	1.4351×10^{-9}	0.0000140625	2.6092×10^{-13},
0.35	0.0000191406	0.00001914219	1.5663×10^{-9},	0.0000191406	3.6981×10^{-13}
0.4	0.0000249999	0.00002500170	1.7086×10^{-9},	0.0000250000	5.0576×10^{-13}
0.45	0.0000316406	0.00003164248	1.8626×10^{-9}	0.0000316406	6.7255×10^{-13}
0.5	0.0000390624	0.00003906452	2.0295×10^{-9},	0.0000390625	8.7442×10^{-13}
0.55	0.0000472656	0.00004726783	2.2102×10^{-9}	0.0000472656	1.1161×10^{-12},
0.6	0.0000562499	0.00005625240	2.4058×10^{-9}	0.0000562500	1.4029×10^{-12}
0.65	0.0000660156	0.00006601824	2.6174×10^{-9}	0.0000660156	1.7406×10^{-12},
0.7	0.0000765624	0.00007656534	2.8464×10^{-9}	0.0000765625	2.1360×10^{-12}
0.75	0.0000878906	0.00008789371	3.0940×10^{-9}	0.0000878906	2.5962×10^{-12}
0.8	0.0000999999	0.00010000336	3.3618×10^{-9}	0.0001000000	3.1295×10^{-12}

Table (4)

Table (4) describes the deference between the error using Trapezoidal and Simpson's rule at time t = 0.01

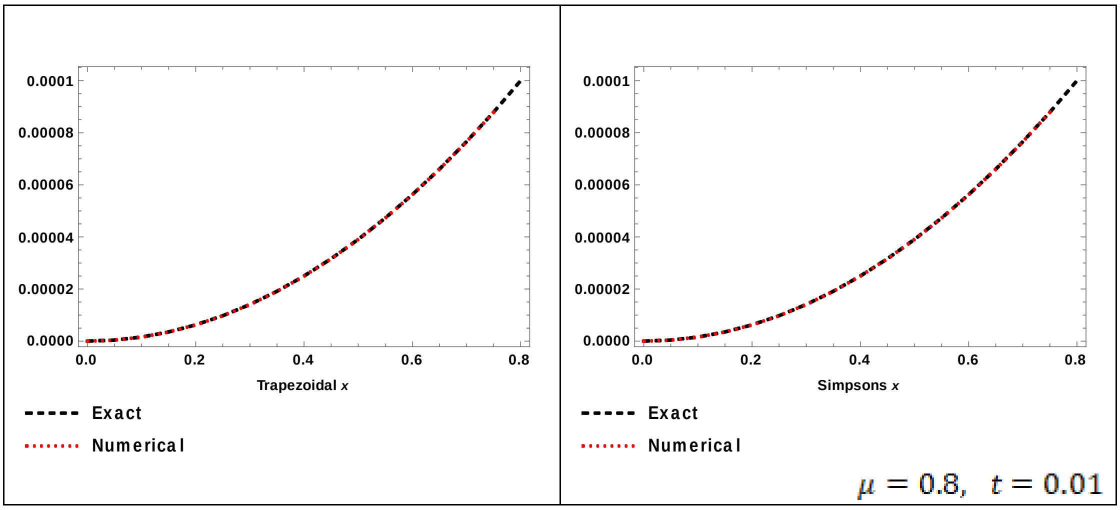

Figure (4)

Figure (4) describe the relationship between exact solution and numerical solution using Trapezoidal and Simpson's rule.

x	Exact Solution	Numerical Solution of Trapezoidal	Error of Trapezoidal	Numerical Solution of Simpson's	Error of Simpson's
0	0	8.3691×10^{-8}	8.3691×10^{-8},	1.4425×10^{-12}	1.4425×10^{-12}
0.05	0.0000390625	0.0000391542	9.1742×10^{-8}	0.0000390625	2.3763×10^{-12},
0.1	0.0001562500	0.0001563504	1.0049×10^{-7}	0.0001562500	5.1533×10^{-12}
0.15	0.0003515625	0.0003516725	1.1001×10^{-7}	0.0003515625	9.9623×10^{-12},
0.2	0.0006250000	0.0006251203	1.2035×10^{-7}	0.0006250000,	1.7017×10^{-11},
0.25	0.0009765625	0.0009766940	1.3158×10^{-7},	0.0009765625	2.6562×10^{-11}
0.3	0.0014062500	0.0014063937	1.4379×10^{-7}	0.0014062500	3.8871×10^{-11}
0.35	0.0019140625	0.0019142195	1.5704×10^{-7},	0.0019140625	5.4253×10^{-11},
0.4	0.0025000000	0.0025001714	1.7143×10^{-7},	0.0025000000	7.3057×10^{-11}
0.45	0.0031640625	0.0031642495	1.8705×10^{-7}	0.0031640625	9.5676×10^{-11}
0.5	0.0039062500	0.0039064540	2.0400×10^{-7}	0.0039062501	1.2255×10^{-10},
0.55	0.0047265625	0.0047267848	2.2239×10^{-7}	0.0047265626	1.5418×10^{-10}
0.6	0.0056250000	0.0056252423	,2.4234×10^{-7}	0.0056250001	1.9112×10^{-10}
0.65	0.0066015625	0.0066018264	2.6398×10^{-7}	0.0066015627	2.3399×10^{-10},
0.7	0.0076562500	0.0076565374	2.8746×10^{-7}	0.0076562502	2.8349×10^{-10},
0.75	0.0087890625	0.0087893754	,3.1291×10^{-7}	0.0087890628	3.4040×10^{-10},
0.8	0.0100000000	0.0100003405	3.4051×10^{-7}	0.0100000004	4.0560×10^{-10}

Table (5)

Table (5) describes the deference between the error using Trapezoidal and Simpson's rule at time t = 0.1

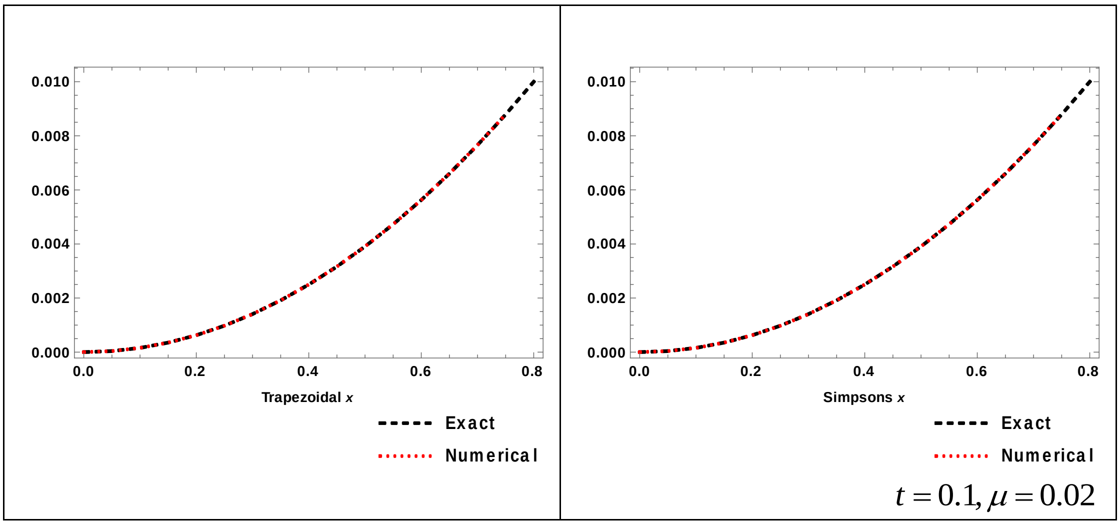

Figure (5)

Figure (5) describe the relationship between exact solution and numerical solution using Trapezoidal and Simpson's rule.

x	Exact Solution	Numerical Solution of Trapezoidal	Error of Trapezoidal	Numerical Solution of Simpson's	Error of Simpson's
0	0	0.0000848304	0.000084830412	0.0000646118	0.00006461187
0.05	0.003525390	0.0034352869	0.000090103668	0.0034566426	0.00006874795
0.1	0.014101562	0.0140054740	0.000096088430	0.0140282310	0.00007333144
0.15	0.031728515	0.0316255170	0.000102998529	0.0316499912	0.00007852433
0.2	0.056406250	0.0562951321	0.000111117894	0.0563217133	0.00008453669
0.25	0.088134765	0.0880139281	0.000120837442	0.0880431120	0.00009165358
0.3	0.126914062	0.1267813467	0.000132715754	0.1268137838	0.00010027869
0.35	0.172744140	0.1725965562	0.000147584393	0.1726331306	0.00011100993
0.4	0.225625000	0.2254582576	0.000166742318	0.2255002214	0.00012477852
0.45	0.285556640	0.2853642982	0.000192342368	0.2854135151	0.00014312542
0.5	0.352539062	0.3523108270	0.000228235447	0.3523702575	0.00016880495
0.55	0.426572265	0.4262902048	0.000282060813	0.4263649860	0.00020727960
0.6	0.507656250	0.5072848219	0.000371428013	0.5073851072	0.00027114270
0.65	0.595791015	0.5952429714	0.000548044192	0.5953936204	0.00039739516
0.7	0.690976562	0.6899195701	0.001056992366	0.6902147197	0.00076184278
0.75	0.793212890	0.7827544002	0.010458490396	0.7849400403	0.00827285025
0.8	0.902500000	0.6949650228	0.207534977124	0.6953052439	0.20719475604

Table (6)

In table (6) we show, the deference between the errors using Trapezoidal and Simpson's rule at time t = 0.95.

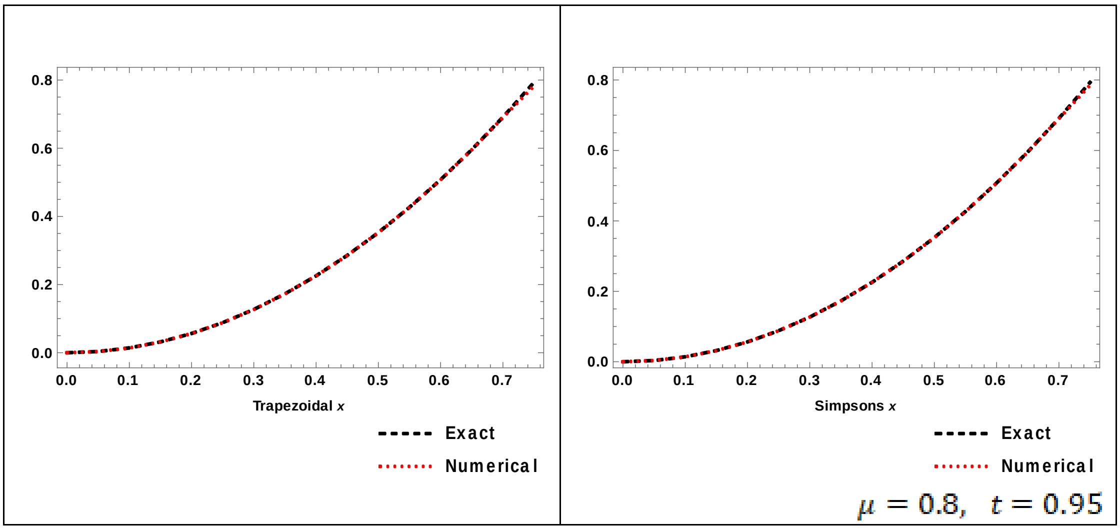

Figure (6)

Figure (6) describe the relationship between exact solution and numerical solution using Trapezoidal and Simpson's rule.

V. CONCLUSION

Here, we consider a nonlinear integral equation of the second kind with continuous kernels, we obtain the numerical solution of the **NF-VIE** using Simpson's method and Trapezoidal method, while, the functions of the integral equations are represented in the form of Simpson's and Trapezoidal rules. The error, in each example is computed.

After using **example 1** Simpson's and Trapezoidal rule we note that:

In table (1) with μ= 0.8 and t = 0.01, we noticed in trapezoidal method and Simpson's method the error decreasing in the period of $0 \le x \le 0.25$ and increasing in the period of $0.3 \le x \le 0.8$.

In table (2) with μ= 0.8 and t = 0.1, as we noticed in trapezoidal method and Simpson's method the error decreasing in the period of $0 \leq x \leq 0.25$ and increasing in the period of $0.3 \leq x \leq 0.8$

In table (3) with μ= 0.8 and t = 0.95, as we noticed in trapezoidal method and Simpson's method the error decreasing in the period of $0 \leq x \leq 0.2$ and increasing in the period of $0.2 \leq x \leq 0.8$.

By using **example 2** for Simpson's and Trapezoidal rule we note that:
In table (4) with μ= 0.8 and t = 0.01, we noticed that the error stable in trapezoidal method, but in Simpson's method the error decreasing in the period of the period of $0 \leq x \leq 0.15$, and decreasing in period $0.2 \leq x \leq 0.8$

In table (5) with μ= 0.02 and t = 0.1, we noticed in trapezoidal method the error is stable in the period of $0 \leq x \leq 0.8$, but in Simpson's method the error decreasing in the period of the period of $0 \leq x \leq 0.15$,and decreasing in period of $0.2 \leq x \leq 0.8$

In table (6) with μ= 0.8 and t = 0.95, we noticed in trapezoidal method the error decreasing in the period of $0 \leq x \leq 0.1$ and increasing in the period of $0.15 \leq x \leq 0.8$ but in Simpson's method the error decreasing in the period of the period of $0 \leq x \leq 0.25$, and decreasing in period of $0.3 \leq x \leq 0.8$

We conclude that we have a good accuracy results with Simpson's method to solve nonlinear Fredholm - Volterra integral equation.

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