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Calculation of Plates on Elastic Foundation by the Generalized Equations of Finite Difference Method

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--***ABSTRACT***--
For the calculation of bending plates on elastic foundation we use the Generalized Equations of the Finite Difference Method. The algorithm allows taking into consideration the finite discontinuities of the desired function, its first derivative and the right part of the differential equation without introduction of fictitious points or a particular tightening of the mesh. The examples presented here show the accuracy of the results and the simplicity of the algorithm.
KEYWORDS: *Plate, Elastic Foundation, Generalized Equations, Finite Difference Method, Boundary Conditions*

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I. INTRODUCTION

The plates on elastic foundation have found wide use in the fields of civil engineering (rafts) and are very effective as a construction element. The calculation of such elements is complicated because it leads to the resolution of partial differential equations with possible variable coefficients. This kind of problem is solved either by analytical methods or by numerical methods. Analytical methods yield accurate results, but are sometimes very complex and difficult to implement [1], [2], [3], [4], [5]. Numerical methods are more efficient, but are approximate [6]. Among the numerical methods, the finite element method is the most widely used [6], [7], [8], [9], [10], [11]. Nevertheless, in this article, we will use the generalized equations of the finite difference method, considering their precision and simplicity. The ultimate goal is to develop an algorithm for calculating plates on an elastic foundation.

II. METHODOLOGY

The steps of the adopted methodology are as follows

- Transformation of the fourth-order partial differential equation of the deflection of a plate on an elastic foundation into a system of two differential equations of second order with partial derivatives
- Introduction of the new dimensionless parameters in the system of equations thus obtained and in the equations describing the boundary conditions;
- Substitution of the new differential equations by the generalized equations of the finite difference method, which makes it possible to obtain a system of algebraic equations;
- Elaboration of a calculation algorithm;
- Solving the system of algebraic equations to get the bending momentum and deflection coefficients.

II.1 Differential equation of the deflection of a plate on an elastic foundation

The differential equation of a thin plate, isotropic in flexion resting on elastic foundation [12], [13] can be reduced to a system of two partial differential equations of second order:

$$\frac{\partial^2 M}{\partial X^2}+\frac{\partial^2 M}{\partial Y^2}=-R; \qquad (1)$$

$$\frac{\partial^2 W}{\partial X^2} + \frac{\partial^2 W}{\partial Y^2} = -\frac{M}{D}, \qquad (2)$$

Where

$$R = P(X,Y) - q(X,Y); \qquad (3)$$

W - the deflection (unknown function); P - load given on the plate; q - reaction due to the pressure of the elastic foundation; $D = \frac{Eh^3}{12(1-\mu^2)}$ – cylindrical (flexional) rigidity of the plate of constant thickness.

$$M = \frac{M_{X+}M_Y}{1+\mu}; \;\; M_X = -D\left(\frac{\partial^2 W}{\partial X^2} + \mu\frac{\partial^2 W}{\partial Y^2}\right); \;\; M_Y = -D\left(\frac{\partial^2 W}{\partial Y^2} + \mu\frac{\partial^2 W}{\partial X^2}\right), \qquad (4)$$

Where : μ- Poisson's ratio; $E -$ Young's modulus; $h -$ thickness of the plate. Equations (1) and (2) are solved by taking into consideration the limit conditions.

II.2 Limit conditions

II.2.1 **Articulated supports**

If the given side is parallel to the X axis, then

$$M_Y = -D\left(\frac{\partial^2 W}{\partial Y^2} + \mu\frac{\partial^2 W}{\partial X^2}\right) = 0; \;\; W = 0 \qquad (4a)$$

If the given side is parallel to the Y axis, then

$$M_X = -D\left(\frac{\partial^2 W}{\partial X^2} + \mu\frac{\partial^2 W}{\partial Y^2}\right) = 0\;; \; W = 0\;; \qquad (4b)$$

II.2.2 **Recessed supports**

If the given side is parallel to the X axis, then

$$\frac{\partial W}{\partial Y} = 0, \quad W = 0 \qquad (5a)$$

If the given side is parallel to the Y axis, then

$$\frac{\partial W}{\partial X} = 0, \quad W = 0 \qquad (5b)$$

II.2.3 **Free sides**

If the given side is parallel to the X axis, then

$$\frac{\partial M}{\partial Y} = 0, \;\; V_Y = 0 \qquad (6a)$$

If the given side is parallel to the Y axis, then

$$\frac{\partial M}{\partial X} = 0, \;\; V_X = 0 \qquad (6b)$$

III. INTRODUCTION OF DIMENSIONLESS PARAMETERS

III-1 Differential equation of the deformed plate shape

Let us Introduce the following parameters [13], [14] :

$$\xi = \frac{X}{a}; \; \eta = \frac{Y}{a}; r = \frac{R}{P_0}; \; m = \frac{M}{P_0 a^2} \;\; ; \; \omega = \frac{WD}{P_0 a^4}; \; m^{(\xi)} = \frac{M_X}{P_0 a^2}; \;\; m^{(\eta)} = \frac{M_y}{P_0 a^2} \qquad (7)$$

Where ξ, η – Cartesian dimensionless coordinates ; a– the length of the smaller side of the rectangular plate P_{0-} a fixed value of the charge P.

Let us write equations (1) and (2) with the new dimensionless parameters (7):

$$\frac{\partial^2 m}{\partial \xi^2} + \frac{\partial^2 m}{\partial \eta^2} = -r; \qquad (8)$$

$$\frac{\partial^2\omega}{\partial\xi^2}+\frac{\partial^2\omega}{\partial\eta^2}=-m, \qquad (9)$$

$$m^{(\xi)}=-\left(\omega^{\xi\xi}+\mu\omega^{\eta\eta}\right);\ \ m^{(\eta)}=-\left(\omega^{\eta\eta}+\mu\omega^{\xi\xi}\right) \qquad (10)$$

$$m^{(\xi\eta)}=-m^{(\eta\xi)}=(1-\mu) \qquad (11)$$

Where :

$$\omega^{\xi\xi}=\frac{\partial^2\omega}{\partial\xi^2};\ \omega^{\eta\eta}=\frac{\partial^2\omega}{\partial\eta^2};\ \omega^{\eta\xi}=\frac{\partial^2\omega}{\partial\xi\partial\eta};\ m^{(\xi\eta)}=\frac{M_{XY}}{P_0a^2}. \qquad (12)$$

Considering the formulas (7) and (9), we can obtain the expression of the dimensionless shear from the corresponding formulas ⌊2⌋ :

$$\frac{T_X}{P_0a}=\frac{\partial m}{\partial\xi}=m^{\xi};\ \frac{T_Y}{P_0a}=\frac{\partial m}{\partial\eta}=m^{\eta}. \qquad (13)$$

III-2 Limit conditions

Let us consider the boundary conditions on the side of the plate corresponding to η = 0 (side parallel to X).

III.2.1 Articulated supports

Equations (4a) taking into consideration (7) and (10) will be written as follows :
$\omega=0$; $m^{(\eta)}=0$. From the first condition, it results $\omega^{\xi\xi}=0$; therefore from formulas (10) and (9) we obtain $m=0$, when we take into consideration the second limit condition.

III.2.2 Embedded supports

Let us write equations (5a) taking into consideration (7) and (10) : $\omega=0$, $\omega^{\eta}=0$; with $\omega^{\eta}=\frac{\partial\omega}{\partial\eta}$.

III.2.3 Free edges

Let us write equations (6a) according to the dimensionless parameters: $m^{(\eta)}=0$, $v^{(\eta)}=0$;
With : $v^{(\eta)}=\frac{V_Y}{P_0a}-$ generalized dimensionless cutting force

IV. SUBSTITUTION OF DIFFERENTIAL EQUATIONS BY THE GENERALIZED EQUATIONS OF THE FINITE DIFFERENCE METHOD

IV.1 Equation of the bending plate

The numerical resolution of the problem will be carried out on regular mesh, so the pitch is h in the direction of the coordinate axes of $\xi\ et\ \eta$. Part of this mesh is shown in Figure 1, where Roman numerals indicate the numbers of elements that have a common point i,j. When m is continuous and $h=\tau$ the approximation of equation (8) is obtained from the equation (2.2.6) [14] replacing ω, p respectively by m,r.

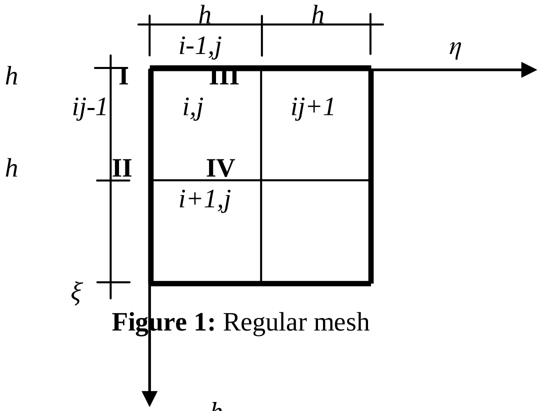

Figure 1: Regular mesh

$$m_{i-1j}+m_{ij-1}-4m_{ij}+m_{ij+1}+m_{i+1j}+\frac{h}{2}\left(\Delta^{I-II}m_{ij}^{\xi}+\Delta^{III-IV}m_{ij}^{\xi}+\Delta^{I-III}m_{ij}^{\eta}+\Delta^{II-IV}m_{ij}^{\eta}\right)$$

$$=-\frac{h^2}{4}\left({}^{I}r_{ij}+{}^{III}r_{ij}+{}^{II}r_{ij}+{}^{IV}r_{ij}\right) \qquad (14)$$

Where :

$$\Delta^{I-II} m_{ij}^{\xi} = {}^{I}m_{ij}^{\xi} - {}^{II}m_{ij}^{\xi}. \quad (15)$$

Other expressions of the same kind are obtained by analogy. Equation (14) is called the generalized equation of the finite difference method, given that if $m^{\xi},\ m^{\eta}\ and\ r$ are continuous, then we obtain as a particular case the equation of the finite difference method which substitutes equation (8)

Since equations (8), (9) differ only in their parameters, then the substitution of equation (9) when $m,\ \omega,\ \omega^{\xi},\ \omega^{\eta}$ are continuous, is obtained by writing equation (14) where m, r are replaced by ω, m respectively.

$$\omega_{i-1j} + \omega_{ij-1} - 4\omega_{ij} + \omega_{ij+1} + \omega_{i+1j} = -h^2 m_{ij}. \quad (16)$$

Note that the partial derivatives of $\omega:\ \omega^{\xi} = \frac{\partial\omega}{\partial\xi}\ et\ \omega^{\eta} = \frac{\partial\omega}{\partial\eta}$ may be discontinuous when the plate has ball joints, while m will be discontinuous if external point bending moments are applied in one of the directions of the coordinate axes. These cases are rare in practice and will not be examined here.

IV.2 Limit conditions

When all the sides of the plate are articulated and the relation between q and W is given, in particular when q = 0 the problem boils down to solving the system of equations (14), (16). These equations will be written for each point inside the integration domain. If the bearing conditions differ from the joints, then the corresponding limit conditions must be substituted.

Suppose the left side of the plate (η = 0) is fixed. To calculate ω_{ij}^{η}, let us write the corresponding equation [12] for a square mesh when $m,\ \omega,\ \omega^{\xi},\ \omega^{\eta}$ are continuous:

$$\omega_{i-1j} - 2h\omega_{ij}^{\eta} - 4\omega_{ij} + 2\omega_{ij+1} + \omega_{i+1j} = -h^2 m_{ij} \quad (17)$$

According to the limit conditions of the embedded sides: $\omega_{ij}^{\eta} = \omega_{i-1j} = \omega_{ij} = \omega_{i+1j} = 0$ and from (17) follows:

$$\omega_{ij+1} = -\frac{h^2}{2} m_{ij} \quad (18)$$

For the upper side of the plate (ξ = 0) equation (17) is written by replacing η, i; j respectively by ξ, j, i. For the right (η = 1) and lower (ξ = 1) sides these equations are written by reflection, but $\omega_{ij}^{\xi},\ \omega_{ij}^{\eta}$ change sign.

Let's examine the case of the free side (η = 0). To determine ω and m at the point ij on the free side of the plate, we use the equations, obtained by substituting the conditions (6a). For that, it is enough to write the equations (10), (15), obtained in [14], [15] when $g = 1,\ g^{\eta} = g^{\xi} = 0$:

$$\omega_{i-1j} - 2\omega_{ij} + \omega_{i+1j} = -\frac{h^2}{1-\mu} m_{ij}\ ; \quad (19)$$

$$\frac{\mu}{2} m_{i-1j} + \frac{1-\mu}{h^2}\left(\omega_{i-1j} - \omega_{i-1j+1}\right) - \left(\frac{3}{2} + \mu\right) m_{ij} + 2m_{ij+1} - \frac{1}{2} m_{ij+2} -$$

$$-\frac{2(1-\mu)}{h^2}\left(\omega_{ij} - \omega_{ij+1}\right) + \frac{\mu}{2} m_{i+1j} + \frac{1-\mu}{h^2}\left(\omega_{i+1j} - \omega_{i+1j+1}\right) = 0. \quad (20)$$

The equation for the point ij of the free upper side of the plate is obtained from equation (20) by replacing i, j respectively by j, i. For the right side and the bottom side these equations are written by reflection. Equation (19) is written by analogy for the different sides.

We notice that the equations (17), (19), (20) describe all the limit conditions according to the unknowns m and ω.

If the plate is not in contact with the ground, then in (14) r will be replaced by $\bar{P} = \frac{P}{P_0}$.

If the plate is in contact with the ground, then $r = \bar{P} + \bar{q}$, the reaction coefficient of the soil is determined according to the elastic modulus of the soil by the formula $\bar{q} = k\omega$, where ω is determined by (7), $k = \frac{Ca^4}{D}$, $C -$ the stiffness of the soil in N/m^3. In that case :

$$r = \bar{P} - k\omega. \qquad (21)$$

Putting (21) in (14) when ω is continuous and k = const, we get the equation that allows to calculate the plate on elastic foundation according to the Winkler model [3], [4], [16].

I. Calculation Algorithm

The algorithm for calculating on a square mesh is as follows. For all the points of the mesh located inside the domain of integration one writes the equations (16) and (14) taking into account of (21). For a plate whose all sides are articulated, these equations are solved simultaneously by considering m = ω = 0 on the edges. In the other cases of the limit conditions one associates with equations (16), (14) either equation (17) or equations (19), (20). The resolution of the equations thus obtained makes it possible to determine m and ω. The values of $\omega^{\xi\xi}$, $\omega^{\eta\eta}$ are calculated by the known formulas of the finite difference method. The values $m^{(\xi)}$, $m^{(\eta)}$ are obtained by formulas (10). To determine $m^{(\xi\eta)}$ by the formula (11), we must compute $\omega^{\xi\eta}$ by the formula obtained by parabolic approximation of the function ω :

$$\omega_{ij}^{\xi\eta} = \frac{1}{4h^2}\left(\omega_{i-1j-1} - \omega_{i-1j+1} + \omega_{i+1j+1}\right) \qquad (22)$$

From the formula (17) one can draw the expression of ω^{η}. By analogy we can determine the expression of ω^{ξ}. To determine m_{ij}^{η}, just write (17) by replacing ω, m respectively by m, r.
Thus digital resolution gives complete results. All parameters of the stress state of the plate are determined.

II. Applications

To illustrate the algorithm's performance let's start by examining the flexion of a plate without an elastic foundation.

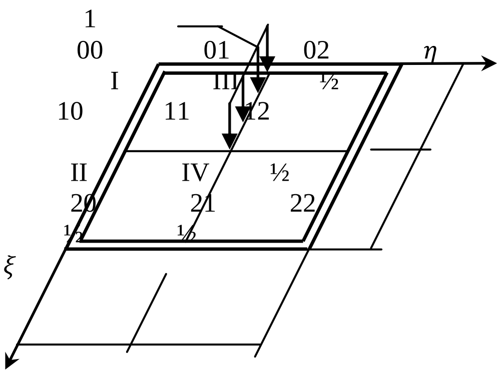

Figure 2 : rectangular plate of constant thickness, with articulated edges

A rectangular plate of constant thickness, all edges of which are articulated, is linearly loaded on line 01-11 as illustrated in figure 2. The intensity of the load is equal to 1. The pitch of the mesh is h = 1/2.
For point 11 let us write equation (14) taking into consideration the limit conditions, when:
$r = \Delta^{I-II} m_{11}^{\xi} = \Delta^{III-IV} m_{11}^{\xi} = \Delta^{II-IV} m_{11}^{\eta} = 0$; $\Delta^{I-III} m_{11}^{\eta} = 1$; $h = 1/2$:
$-4m_{11} + \frac{1}{2}.\frac{1}{2}.1$; $m = \frac{1}{16}$. When the plate is loaded all the way 01-11-21 (figure 2), then $m_{11} = \frac{1}{16}.2 = \frac{1}{8}$. To determine ω_{11} write equation (16) taking into consideration the limit conditions and the value of m_{11}:

$$-4\omega_{11} = -\frac{1}{2^2}.\frac{1}{8}; \quad \omega_{11} = 0.00781$$

We must say that the complexity of loading and rarity of the grid have led to a poor result. When we compare the result of ω_{11} with that obtained in [12] using Fourier's double series, the difference is 16%. If we reduce the mesh size h = 1/4, this difference decreases and becomes equal to 8.6%. Thus by decreasing the pitch of the mesh we can substantially improve the result (Table 1) shows that when h = 1/32 the difference is of the order of 0.15%.

It should be noted that the algorithm was developed for the purpose of digitizing the computation by the generalized equations of the finite difference method. The examples discussed here simply illustrate the effectiveness of the algorithm and show that with a fairly large mesh size the generalized equations of the finite difference method give satisfactory results.

Tableau 1 : Values of the coefficients of momentum and the arrow in the center of the plate

the mesh step size h	Bending momentum in the center of the plate	deflection in the center of the plate
	m	υ
h =1/4	0.125	0.00781
h =1/8	0.166	0.00692
h =1/16	0.168	0.00679
h =1/32	0.168	0.00675
[10]	-	0.00674

Now let's look at a rectangular plate on an elastic foundation. The plate is articulated on the upper and lower sides, while the right and left sides are free. The plate is in contact with the ground k = 200; μ = 0.3 and is subjected to a load uniformly distributed over its entire surface $\bar{P} = 1$. Under the symmetry property, we will work on a part of the plate figure 3.

For points 11, 12 write equations (14), (15) taking into consideration (21), limit conditions and symmetry, when h = 1/2. For point 10, write equations (19), (20). From the resolution of the system of equations thus obtained we determine the parameters: $m_{10} = 0.01944$; $m_{11} = 0.02546$; $m_{12} = 0.03052$; $\omega_{10} = 0.003472$; $\omega_{11} = 0.003352$; $\omega_{12} = 0.003583$.

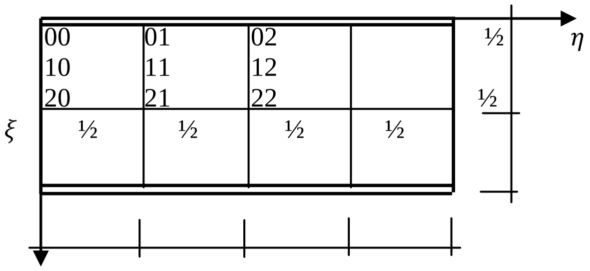

Figure 3: rectangular plate (b = 2a) with two parallel articulated edges and two parallel free edges

Since solutions for this problem do not exist anywhere, we checked the error of the results obtained by using the principle of static equilibrium of the plate. In this view we have determined the sum of the projections of all the reactions on the axis perpendicular to the average plane of the plate. Under the symmetry property, we can consider half of the plate. The resultant of external loads applied to this portion is equal to 1. The reaction at point ij of the articulated side can be determined by the formula that follows from (19) if we write it in agreement with (8) by replacing ω, m, η, i, j respectively by m, r, ξ, j i. When we consider that $r_{ij} = \overline{P_{ij}} - k\omega_{ij}$ and that on the sides $\omega = m = 0$ we obtain:

$$m_{ij}^{\xi} = \frac{1}{2}.\frac{1}{2} + 2m_{ij}. \tag{23}$$

Using the formula (23) determine the reactions on the sides of the plate taking into consideration the values of m calculated above : $m_{00}^{\xi} = 0.2889$; $m_{01}^{\xi} = 0.3009$; $m_{02}^{\xi} = 0.3110$. Due to symmetry $m_{20}^{\xi} = m_{00}^{\xi}$; $m_{21}^{\xi} = m_{01}^{\xi}$; $m_{22}^{\xi} = m_{02}^{\xi}$. The sum of the projections of these reactions on the axis perpendicular to the average plane of the plate is equal to : $\sum m^{\xi} = 0.6012$. Then, using Simpson's formula we determined the reactions at the points of contact with the ground, whose sum of projections on the axis perpendicular to the mean plane of the plate is equal to : $\sum q = 0.4547$. So we get the resulting reaction $R_0 = \sum m^{\xi} + \sum q = 1.056$. From the results obtained, it results that the error is 4.9%.

V. CONCLUSION

In conclusion we can say that the algorithm developed here solves the problems of calculating isotropic thin plates of constant thickness resting on an elastic foundation.

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