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Application of Lambert Functions in Control of Production Systems with Delay

Rudolf Pušenjak *
Faculty of Industrial Engineering Novo mesto Šegova ulica 112, SI-8000 Novo mesto, Slovenia
*Corresponding author: Rudolf Pušenjak **

ABSTRACT

The paper treats the control of production process with production delay. A simple production system with delay is analyzed, where the production of a single product is continuously performed. The Lambert functions are used to solve the governing delayed differential equation of the production process in an analytical form. In the paper, the stability of the production system control is investigated and an analytical stability bound is obtained, whichdecides if the production process can be controlled in the stable area. Analytical results of production process with delay of various size are computed, which reveal an aperiodic or oscillatory character of actual inventory, respectively.Results, obtained by using Lambert function are compared with solutions, computed by Runge-Kutta numerical integration. All comparisons show a perfect matching of both solutions.

Key words:*production system with delay, Lambert functions, stability bound, oscillatory and aperiodic response.*

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I. INTRODUCTION

This paper treats a production process with its inherent property that every production demands some time and introduces some time delay. Time delays, of course, appear also in many physical and engineering systems, for example in the control of nuclear reactors, as transport delays in combustion processes, in the control of robots and manipulators, in chemical reactions and so on. However, while time delays in physics and engineering often can be neglected by compelling reasons, this cannot be applied in the case of the production processes. Ignoring the time delays in production would lead to oversimplification, where the most important phenomena are overlooked. The presence of time delays is the cause of many phenomena, which cannot be expected in systems without delays.The evolution of systems with time delays can be described in the form of delayed differential equations (DDE's). The systems of DDE's in general can be nonlinear (NDDE's) or linear (LDDE's), where the later will be exclusively treated in this paper. For nonlinear systems is advantageous, if the problem at hand can be linearized and methods of solving LDDE's can be applied. If this is not the case, the system ofNDDE's must be solved numerically. Often the integration of NDDE's is performed by Runge-Kutta methods, which is applied also in this paper for the sake of comparisons with analytical solutions. Albeit numerical methods of solving DDE's can be applied in nonlinear as well as in linear systems, this advantage goes on the cost of generality of solutions. Numerical solutions can be obtained only for specified values of parameters. To overcome this deficiency, analytical methods for solving LDDE's are developed in the past two decades, which offer a systematic exploration of the properties of delayed systems.

Among analytical methods of solving LDDE's, the method of Lambert functions [1], [7] has been enforced, which is presented in this paper. In the proposed method, the solution of the system of LDDE's is sought in the form of linear combination of Lambert oscillation modes [1]. Besides the method of Lambert functions, the well knownanalytical method of steps [7] and the classical Laplace transform method are applicable. The deficiency of the method of steps is the lack of transparency and the complex computing, when the number of steps increases. The applicability of the Laplace transform method is limited by implementation of the inverse transformation. For solving LDDE's, various combinations of analytical and numerical methods can be alternatively used, where is characteristic that the original problem is reduced on the numerical solving of the some (easier) algebraic problem. Among these methods the least squares method [7], the Pade approximation [10], the homotopic perturbation method [6], the Adomian decomposition method [5], the conversion on the nonlinear (transcendental) eigenvalue problem [11] and the use of the equivalent integral equations with delay [2] are often applied. Due to the production delays, the inventory can have an oscillatory character, which adversely affects the variable costs of production. The purpose of this paper is to analyze the inventory oscillations in an analytical form, to conquer an in-depth understanding about the control of the production process with delay and to investigate its stability.

II. A SYSTEM APPROACH TO THE CONTROL OF THE PRODUCTION PROCESS WITH DELAY

Time delays are the characteristic property of each production process and the cause of many phenomena, which cannot be expected in processes without delays. In this paper, the simple production process with the time delay is treated, which can be described in the form of LDDE. The governing LDDE of production process will be solved by using Lambert functions [1], [4], [8]. In derivation of the governing LDDE is assumed, that the production consists of manufacturing a single product, which is continuosly produced. The production is managed in such a way that the optimal quantity of product is manufactured.The production is carried out in accordance with the standard regulations and then stored until it is sent to the customers on the basis of their orders. The control of the quantity of the product, which must be daily produced (or must be produced in the other appropriate time unit) is performed by means of production orders.

The goal of the control of production system is the reduction of production costs in the selected time period on the minimum. The variable part of production costs depends on variations of quantity of production as well as on the inventory level of finished products. Production costs per unit of finished product increases if the quantity of production varies in comparison with costs of the constant quantity of production. On the other side, the transport costs increase when the inventory level of finished products increases and the reduction of the stock level of manufactured products belowa certain limit causes a delay in the performance of orders.In order that the production can be controlled, functions of variations of production quantity and the stock level of finished products are introduced. The block diagram of the control of the production system based on the system analysis is shown in Fig. 1.

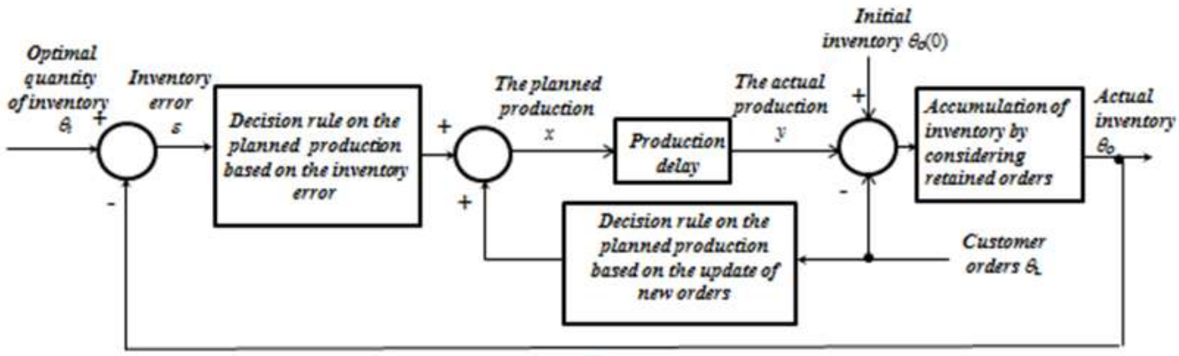

Figure 1. Thesystem approach to the control of production process with time delay

The input variable ofthe controlled production system depicted in the Fig. 1 is the desired quantity of inventory $\theta_i(t)$, which is assumed to be optimal. The actual quantity of inventory of finished products $\theta_o(t)$ represents the output variable of the production system. It is assumed that the actual quantity of inventory of finished products can take positive as well as negative values. The negative inventory level in the fact represents unsatisfied or retained orders of customers, respectively.The control of the quantity of the production is based on the comparison of the optimal quantity of inventory $\theta_i(t)$ with the actual inventory level $\theta_o(t)$, which is expressed by means of positive (or negative) deviation of inventory or so called control error $\varepsilon(t)$:

$$\varepsilon(t) = \theta_i(t) - \theta_o(t) \qquad (1)$$

Customer orders per unit of time are treated as the load of the production process and are denoted by $\theta_L(t)$. The initial quantity of inventory of finished products at the beginning of the production process in Fig. 1 is denoted as $\theta_o(0)$ and is in the rule equal to zero.

The production system can be completely described by means of two additional variables $x(t)$ and $y(t)$, respectively, where $x(t)$ represents the new planned production per unit of time and $y(t)$ represents the actual production per unit of time. The new planned production is determined on the basis of production orders, where it is assumed, that production orders are issued continuously. If the control error $\varepsilon(t)$ is positive, the production order must ensure the increase of the new planned production for the amount $K_1\varepsilon(t)$. In addition, the proportional rate $K_2\theta_L(t)$ must be considered with which new orders of customers are taken into account. Accordingly, the new planned production is described by the relationship:

$$x(t) = K_1\varepsilon(t) + K_2\theta_L(t) \qquad (2)$$

When the daily production order about the new planned production is issued, some time elapses before the required quantity of products is manufactured. The elapsed time represents the delay T between the output of the production order for the new planned production and their realization. The time delay T can be variable in the general setting, however due to the simplicity, it will be considered as constant during the whole production process. The actual production per the time unit takes this delay into account and is expressed by means of the delayed new planned production as:

$$y(t) = x(t - T) \quad (3)$$

The status of the actual inventory $\theta_o(t)$ in the warehouse is represented by means of the accumulated quantity of products less the accumulated amount of customer orders and is expressed in a mathematical form by the following integral:

$$\theta_o(t) = \theta_o(0) + \int_0^t \left[y(\tau) - \theta_L(\tau) \right] \mathrm{d}\tau, \quad (4)$$

where the initial status of inventory in the time t=0 is considered as $\theta_o(0)$. By means of Eqs. (1)-(4) the complete block diagram of the control of production process with delay is drawn up, where the links between particular intermediate variables are determined via transfer functions as are usually used in the control engineering. For example, transfer functions K_1 and K_2 correspond to two proportional controllers, the transfer function e^{-Ts} represents the Laplace transform of the delay block with the time delay T, the transfer function $1/s$ is the Laplace transform of an integrator, which corresponds to the relation (4) and s is the Laplace variable in the complex domain. The block diagram of the control of the production process with delay, which is described by means of input-output relations (1)-(4) is shown in the Fig. 2.

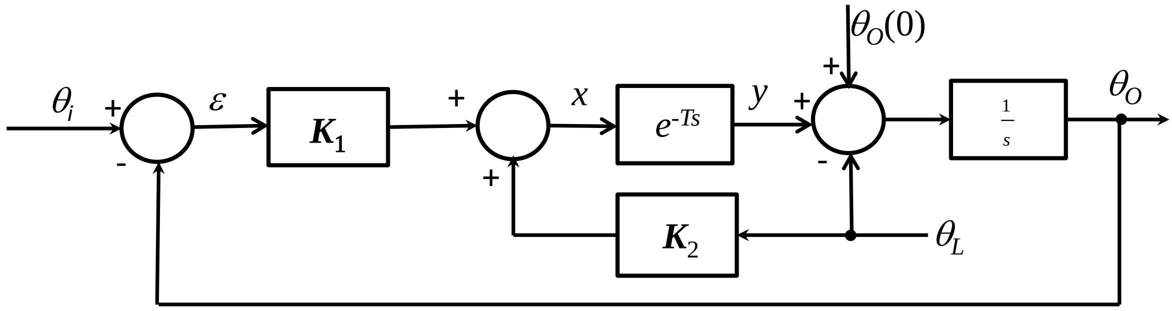

Figure 2. The feedback control of the production process with delay according to the input-output relations (1)-(4).

The accumulation of the inventory in the warehouse, which is expressed by means of the definite integral (4) can be transformed by differentiation on the time t into the more convenient form of differential equation:

$$\frac{\mathrm{d}\theta_o(t)}{\mathrm{d}t} = y(t) - \theta_L(t), \quad (5)$$

By elimination of intermediate variables $\varepsilon(t)$, $x(t)$ and $y(t)$ from the sytem of equations (1), (2), (3) and (5), the governing nonhomogenous LDDE of the production control with delay T is derived:

$$\frac{\mathrm{d}\theta_o}{\mathrm{d}t} + K_1\theta_o(t - T) = K_1\theta_i(t - T) + K_2\theta_L(t - T) - \theta_L(t), \quad (6)$$

where K_1 denotes the rate of planned new production based on the information about control error $\varepsilon(t)$ and K_2 denotes the rate of planned new production considering the information about customer orders. In solving LDDE (6), without loss of generality can be assumed, that the optimal quantity of inventory $\theta_i(t\text{-}T)$ on the preshape time interval [-T,0] is equal zero, $\theta_i(t\text{-}T)$=0, because it appears additively in Eq. (6). By considering this simplification, the following LDDE is obtained:

$$\frac{\mathrm{d}\theta_o}{\mathrm{d}t} + K_1\theta_o(t - T) = K_2\theta_L(t - T) - \theta_L(t). \quad (7)$$

This type of the LDDE is solved in the sequel, where $\theta_L(t)$ is the loading of the system, which is assumed to be known on the time interval [0,t), where the solution of the actual inventory of products $\theta_o(t)$ is sought, as well as is prescribed on the preshape interval [-T,0] in the form of the time dependent function $\theta_L(t\text{-}T)$.

III. SOLVING THE PROBLEM OF THE CONTROL OF PRODUCTION PROCESS WITH DELAY BY MEANS OF LAMBERT FUNCTIONS

When are solving LDDE (7), similarly as in the case of ordinary linear differential equations (OLDE's) at first must be solved the corresponding homogeneous equation. In the case of the control of the production process with delay as is described in the previous section, the homogeneous LDDE of the first order reads as follows:

$$\frac{\mathrm{d}\theta_o(t)}{\mathrm{d}t} + K_1\theta_o(t - T) = 0, \quad T > 0, \quad (8)$$

where the time course of the actual inventory $\theta_o(t)$ on the preshape interval [$-T$,0] is prescribed as the known function $\theta_o(t) = \phi(t)$, $t\in[-T,0]$. From the behaviour of ordinary differential equations (ODE's) is known, that the solution of Eq. (8), when the time delay T is equal zero, T=0, can exponentially grow or can decay. It is usually not expected, that the solution of Eq. (8) can have an oscillatory character or is even unstable at delays, which are great enough, $T>0$. It will be shown later, that the nature of the particular solution is strongly dependent on the relationship between values of parameters K_1 and T as well as of the prescribed function $\phi(t)$ on the preshape interval.
The ansatz for solution of Eq. (8) has the same form $\theta_o(t) = Ce^{st}$ as for ODE's. By substitution of the ansatz into Eq. (8), one obtains the nonlinear transcendental characteristic equation:

$$F(s) = se^{sT} + K_1 = 0 \quad (9)$$

The difficulty to obtain an analytical solution of LDDE originates from the fact, that an analytical method of solving of the transcendental equation is needed.
With the purpose to solve Eq. (9), a new approach is presented, which is based on the special function $W(s)$, which is called the Lambert function. On the basis of definition [4], the Lambert function is called such a function, which satisfies the equation

$$W(s)e^{W(s)} = s\,, \quad (10)$$

This definition can be used for solving the Eq. (9), which is at first rewritten in the form:

$$sTe^{sT} = -K_1T\,, \quad (11)$$

and then Eq. (10) is used on the argument $-K_1T$:

$$W(-K_1T)e^{W(-K_1T)} = -K_1T\,. \quad (12)$$

Equation (11) now can be written as follows:

$$sTe^{sT} = W(-K_1T)e^{W(-K_1T)}\,, \quad (13)$$

which holds only then, when it holds:

$$sT = W(-K_1T)\,. \quad (14)$$

From Eq. (14) one obtains the equation, which can be applied for computing roots of the characteristic Eq. (9):

$$s = \frac{1}{T}W(-K_1T)\,, \quad (15)$$

where individual roots are expressed in terms of Lambert function.
Lambert function in general is a complex function, which has an infinite number of branches. The branches of Lambert function are denoted as $W_k(s)$, where the index k takes the values $k= -\infty,\dots,-2,-1,0,1,2,\dots,+\infty$. The index k=0 belongs to the fundamental branch of the Lambert function, which is denoted as $W_0(s)$. The fundamental branch takes real values on the interval [$-1/e,+\infty$), while it is complex outside this interval. The fundamental branch of Lambert function is computed by means of the series:

$$W_0(s) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!}s^n \quad (16)$$

(Caratheodory [3]). Outside of the interval [$-1/e,+\infty$) the Lambert function $W_0(s)$ is complex. The branch $W_{-1}(s)$ with index $k = -1$ is real valued on the interval [$-1/e$,0), but complex outside of this interval. All other branches $W_k(s)$ of Lambert function, where the index k takes values $k = -\infty,\dots,-2$ or $k = 1,2,\dots,+\infty$, respectively, are complex on the whole domain of the argument s and can be computed by means of the formula [3]:

$$W_k(s) = \ln_k(s) - \ln\left(\ln_k(s)\right) + \sum_{l=0}^{\infty}\sum_{m=1}^{\infty} C_{lm}\frac{\left(\ln\left(\ln_k(s)\right)\right)^m}{\left(\ln_k(s)\right)^{l+m}}\,, \quad (17)$$

The function $ln_k(s)=ln(s)+2\pi ik$ is the k-th logarithmic branch, where $i = \sqrt{-1}$ is the imaginary unit and C_{lm} are coefficients, which means Stirling cyclic numbers:

$$C_{lm} = \frac{1}{m!}(-1)^l\begin{bmatrix} l+m \\ l+1 \end{bmatrix} \quad (18)$$

Lambert functions do not belong to the standard functions and thus cannot be computed by scientific calculators. However, various software developers recognized the meaning of Lambert functions in different areas, what led to the implementation of symbolic computation of Lambert functions at arbitrary values of the argument $s$within programming environments such as Maple, Mathematica and Matlab, respectively. Moreover, all these programming systems allow the execution of various symbolic operations with Lambert functions, such as differentiation or integration, etc. In the programming system Matlab, the Lambert function can be computed by the function call named *LambertW*, while in Mathematica the same task can be done by the function call of *ProductLog*[*k*,*s*], where index *k* denotes the branch of Lambert function and the symbol *s* means their argument. For example, in the programming system Mathematica one computes and shows easily the course of the fundamental branch of Lambert function $W_0(s)$ for real values of the argument *s* in the interval $[-1/e,+\infty)$ as well as the course of the branch $W_{-1}(s)$ for real values of the argument *s* in the interval [-1/e,0), respectively, by means of the following program:

```
graph1 = Plot[ProductLog[0, x], {x, -1 / E, 1}];
graph2 = Plot[ProductLog[-1, x], {x, -1 / E, 0}, PlotStyle → {RGBColor[1, 0, 0], Dashed}];
Show[graph1, graph2, PlotRange → All, AxesLabel → {s, "  W₀(s)  , W₋₁(s)"}]
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The result of computation is depicted in Fig. 3:

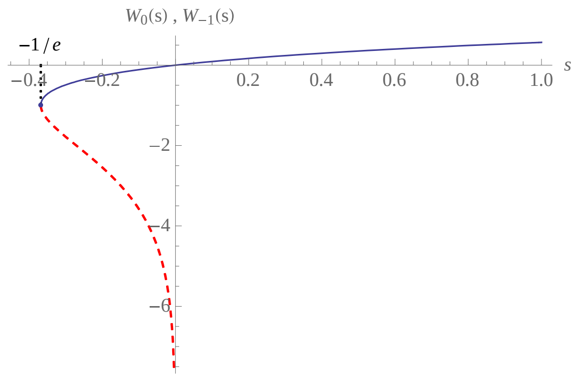

Figure 3. Plot of real valued parts of branches $W_0(s)$ and $W_{-1}(s)$.

From the above diagram it can be seen, that fundamental branch of Lambert function satisfies inequality $-1 \leq W_0(s)$ on the entire interval $-1/e \leq s < +\infty$, while the branch $W_{-1}(s)$ satisfies the inequality $W_{-1}(s) \leq -1$ on the interval $-1/e \leq s < 0$.

As an example we look on the computation of first 31 roots of the characteristic equation (9), if values of parameters $K_1=T=1$ are chosen. These roots are computed by means of Eq. (15). The equation is for the chosen data $K_1=T=1$ reduced on equation $s_k=1\times W_k(-1)=W_k(-1)$, where the index k takes integer values from -15 until +15. The position of the first 31 characteristic roots in the complex (Gauss) plane shows the Fig. 4.

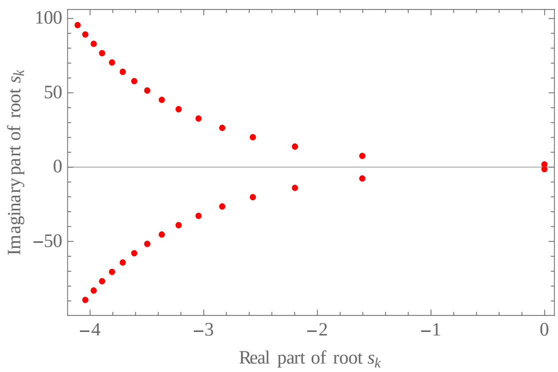

Figure 4. The position of roots of the transcendental characteristic equation in the complex plane.

As we can see on this plot, all roots $W_k(-1)$, k=-15,…,15 are complex and have besides nonzero real part also an imaginary part. This property is the consequence of the fact, that the argument $s = -1$ of Lambert function is less than -1/e = -0.367879, which is the value of lower bound of the interval, where Lambert functions W_0 and W_{-1} are real valued. Roots $W_{-1}(-1)$ and $W_0(-1)$ in the above diagram have smallest imaginary part among all roots, which are shown in the Fig. 4 and are complex conjugate to each other: $W_{-1}(-1) = -0.318132-1.33724i$, $W_0(-1) =-0 .318132+1.33724i$. Roots $W_{-1}(-1)$ and $W_0(-1)$ lie in the Gauss plane at smallest distance to the imaginary axis and determine the stability of the system. If we would have characteristic roots $W_0(\cdot)$ in $W_{-1}(\cdot)$ with positive real parts, the system would be unstable. From this it follows, that the bound between stable and unstable area is determined by means of characteristic roots $W_0(\cdot)$ and $W_{-1}(\cdot)$, which have their real part equal zero and at the same time a nonzero imaginary part. As can be seen in the Fig. 3, the fundamental branch $W_0(\cdot)$ has the value $W_0(0) = 0$ at the coordinate origin, that is for the value of the argument $s = 0$. The value $W_0(0) = 0$ cannot be taken into account for determination of stability bound, because doesn't satisfy the condition, that characteristic root must have a zero real part and at the same time a nonzero imaginary part.

By solving Eq. (15) in programming envinronment Mathematica (for example, by using the command FindRoot[Re[ProductLog[0,-*KT*]],{ *KT*,1/E}], where the index $_1$ in K_1 is omitted due to the syntax rules), we can find the solution $K_1T = \pi/2$. The obtained solution represents an equation of hyperbola, which determines the bound between stable and unstable area, when parameters K_1 and T vary. The obtained bound is based on the stability criterium of the branch $W_0(-K_1T)$. In a similar way we can seek the value K_1T of the branch $W_{-1}(\cdot)$, where the Lambert function $W_{-1}(\cdot)$ is pure imaginary and its argument lies outside the interval $[-1/e,0)$, as is clear from the Fig.3. To solve this task, we apply commands FindRoot[Re[ProductLog[-1, -*KT*]],{ *KT*,1/E}] and FindRoot[Re[ProductLog[-1,- *KT*]],{ *KT*,*tol*}], respectively, where *tol* denotes a small positive number. Both commands return the same value $K_1T = \pi/2$ as is obtained earlier at the branch $W_0(\cdot)$. Values of product K_1T for other roots of Eq. (15), which have the real part equal zero and at the same time the nonzero imaginary part, however differ from the value of $\pi/2$. For example, we obtain the value $K_1T = 5\pi/2$ for branches $W_{-2}(\cdot)$ and $W_1(\cdot)$, the value $K_1T = 9\pi/2$ for branches $W_{-3}(\cdot)$ in $W_2(\cdot)$, and so on. Hyperbolic courses at first three pure imaginary roots of Eq. (15) with indexes k=0,1 and 2 (what means that each root has a zero real part and at the same time the nonzero imaginary part) are shown in the Fig. 5. Hyperbola, which corresponds to the branch $W_0(-K_1T)$, represents the stability bound of the system, which is not destroyed by other hyperbolas. Bellow the hyperbola of the branch $W_0(-K_1T)$ with the pure imaginary root is the stable area and above the hyperbola is the unstable area. From the diagram on the Fig. 5 it follows, that at fixed value of the time delay T we can pass from the stable area into unstable area only by increasing the parameter K_1. Similarly, at fixed value of parameter K_1 we can pass from the stable area into unstable area only by increasing the time delay T.

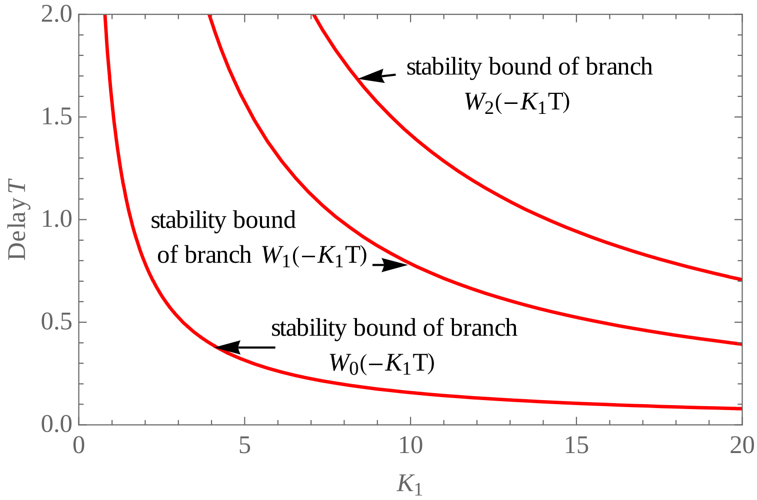

Figure 5. Stability bounds of several branches of Lambert function in the control of production system with delay

The general solution of homogeneous Eq. (8) with computed roots of Eq. (15) can now be written in the form:

$$\theta_o(t) = \sum_{k=-\infty}^{\infty} C_k e^{\left[\frac{1}{T}W_k(-K_1T)\right]t} = \sum_{k=-\infty}^{\infty} C_k \xi_k(t)\,, \quad (19)$$

where C_k are coefficients, which are determined by means of the prescribed function $\phi(t)$ on the preshape interval $[-T,0]$. Functions

$$\xi_k(t) = e^{\left[\frac{1}{T}W_k(-K_1T)\right]t} \quad (20)$$

are named Lambert oscillation modes and coefficients C_k are called Lambert coefficients. For practical computing one consider a finite number of dominant Lambert oscillation modes, which is high enough and approximate the infinite series in Eq. (19) by the truncated series:

$$\theta_o(t) = \sum_{k=l}^{N} C_k \xi_k(t) \quad (21)$$

where number N approaches infinity, $N\to\infty$ in the theoretical sense. In order to assure real solution of the homogeneous Eq. (8), index l must be equal to $l = -(N+1)$ in the case of asymmetric numbering of branches of Lambert function and to $l = -N$ at symmetric numbering of branches, respectively. Asymmetric numbering of branches of Lambert function must be applied in the case of the negative argument of Lambert function, that is, when $-K_1T < 0$. From the Fig. 3 it is evident, that in the case of the negative argument in addition to the fundamental branch $W_0(-K_1T)$ also the branch $W_{-1}(-K_1T)$ must be taken into account in the series (21), while in the case of the positive argument only the branch $W_0(-K_1T)$ must be considered. That explains the difference in choosing of counter l at asymmetric and symmetric numbering of branches.

3.1. Determination of coefficients C_k in the general solution of the homogeneous equation

In common an arbitrary given continuous function can be expressed in the form of series (19). Thus, by appropriate choice of Lambert coefficients C_k and corresponding Lambert oscillation modes $\xi_k(t)$ we can express the prescribed function $\phi(t)$ on the preshape interval [$-T$,0] in the form:

$$\phi(t) = \sum_{k=-\infty}^{\infty} C_k \xi_k(t), \quad t \in [-T, 0]. \qquad (22)$$

From the practical reasons, the infinite series in Eq. (22) must be truncated also in this case and replaced by the finite series:

$$\phi(t) = \sum_{k=l}^{N} C_k \xi_k(t), \quad t \in [-T, 0]. \qquad (23)$$

In similar way as in the case of ordinary differential equations, we can use Eq. (23) in the reverse sense to determine unknown Lambert coefficients C_k by the help of known values of function $\phi(t)$. The number of unknown coefficients C_k is equal to $2N+1$ in the case of symmetric numbering of branches of Lambert function and to $2N+2$ in the case of asymmetric numbering. Accordingly, the entire interval $[-T,0]$ is divided into $M=2N$ subintervals of equal length at symmetric numbering or $M=2N+1$ subintervals of equal length at asymmetric numbering, respectively. By using such a division of the interval $[-T,0]$, the prescribed function $\phi(t)$ as well as Lambert oscillation modes $\xi_k(t)$ are sampled at time instants $-T, -T+\frac{T}{M}, -T+\frac{2T}{M}, \cdots, \frac{T}{M}, 0$, where corresponds to each sample an equation of the form:

$$\phi\left(-T+\frac{iT}{M}\right) = \sum_{k=l}^{N} C_k \xi_k\left(-T+\frac{iT}{M}\right), \quad i = 0,1,\cdots,M \qquad (24)$$

The number of Eqs. (24), which are created in such manner, is equal to $M+1$ and corresponds to $2N+1$ coefficients at symmetric numbering of branches, or corresponds to $2N+2$ coefficients in the case of asymmetric numbering. The system of equations (24) is conveniently written in the matrix form as follows:

$$\mathbf{\Phi} = \mathbf{\Xi} \cdot \mathbf{C} \qquad (25)$$

Under assumption, that inverse matrix of Lambert oscillation modes □$^{-1}$ exists, we can compute the unknown vector of Lambert coefficients **C** by means of equation **C**= □$^{-1}$·□. An individual Lambert coefficient C_k can be obtained by scalar multiplication of the k-th row of matrix □$^{-1}$ with the vector □:

$$C_k = \lim_{N \to \infty} \left(\mathbf{\Xi}^{-1} \cdot \mathbf{\Phi}\right)_k, \quad k \in \{l, l+1, \cdots, N\}. \qquad (26)$$

By applying Eq. (26), we can write solution (21) of the homogeneous LDDE in the following form:

$$\theta_o(t) = \sum_{k=l}^{N} \lim_{N \to \infty} \left(\mathbf{\Xi}^{-1} \cdot \mathbf{\Phi}\right)_k \xi_k(t) = \sum_{k=l}^{N} \lim_{N \to \infty} \left(\mathbf{\Xi}^{-1} \cdot \mathbf{\Phi}\right)_k e^{\left[\frac{1}{T} W_k(-K_1 T)\right] t}. \qquad (27)$$

3.2. The formation of complete solution of nonhomogeneousLDDE

Now return to the formation of the complete solution of nonhomogeneous differential equation of first order with delay, that is to the solving of Eq. (7). The complete solution of nonhomogeneous LDDE is obtained by adding some particular solution to the solution of homogeneous differential equation with delay, derived in this paperin the form of Eq. (21) and then by considering the practical aspects of computation in the form of Eq. (27).Computing of the particular solution of LDDE can be performed on many different ways so that the interested reader is advised on the furtherreferences [1], [2],[7], [9].In this paper, the particular solution of Eq. (7) is sought in the form of integral equation:

$$\theta_o^P(t) = \int_0^t \psi(t,\zeta)\, b u(\zeta)\, d\zeta = \int_0^t \sum_{k=l}^{\infty} C_k^P e^{\left[\frac{1}{T} W_k(-K_1 T)\right](t-\zeta)} b u(\zeta)\, \mathrm{d}\zeta, \quad t \ge 0, \qquad (28)$$

where $\psi(t,\zeta)$ is the kernel of integral equation, $bu(\zeta)$ is ageneralized form of the right hand side of Eq. (7) and C_k^P are unknown coefficients of the particular solution.The complete solution of nonhomogeneous equation (7) then can be written in the following form:

$$\theta_o(t) = \sum_{k=l}^{N} C_k e^{\left[\frac{1}{T}W_k(-K_1 T)\right]t} + \int_0^t \sum_{k=l}^{N} C_k^P e^{\left[\frac{1}{T}W_k(-K_1 T)\right](t-\zeta)} bu(\zeta)\,\mathrm{d}\zeta\ , \qquad (29)$$

where unknown coefficients C_k^P of the particular solution are determined by means of the Laplace transformation and theory of residues in the compact form:

$$C_k^P = \frac{1}{1-K_1 T e^{-W_k(-K_1 T)}}\ . \quad (30)$$

IV. RESULTS AND DISCUSSION

Now look on the solution of the control of production process with delay in which retained orders of customers are taken into account. In other words, we look on the evolution of the actual inventory $\theta_o(t)$ in the warehouse as is described by Eq. (7). For the sake of simplicity, we assume parameter valuesK_1=K_2=1 and the time delay T=1, respectively, and choose the time course of customer orders in the form of step function, what means, that customer orders are constant for all times t>0 .Solving of the possed problem in the programming envinronment Mathematica doesn't represent any difficulties, just the function of the system loading b□ $u(t)$=b□UnitStep[t] must be replaced in Eq. (29) by the function b_1□$u(t)$+ b_2□$u(t-T)$ = b_1□UnitStep[t] +b_2□UnitStep[t-T], where $b_1 = -K_1 = -1$ and $b_2 = K_2 = 1$.The solution by using only 2 branches of Lambert function is shown in Fig. 6, where 2 branches corresponds to the number N=0.Despite such a rough approximation a pretty good match with a numerical solution by using Runge-Kutta method can be seen in Fig. 6 on the whole time interval [$-T$,10]. The response $\theta_o(t)$ shows an oscillatory character with positive and negative half-periods on the interval [0,10], where negative values have meaning of unsatisfied orders and positive values correspond to the amount of actual inventory. Oscillation amplitudes of actual inventory gradually decrease on the whole interval [0,10] against the zero value.Of course is zero value of actual inventory theoretically reached after an infinite time, but this stationary state is important, because it can be interpreted as a state when unsatisfied orders are abolished and actualy inventory disappear.The stationary value of actual inventory can be analytically calculated, if the limit value $\lim_{t\to\infty} \frac{\mathrm{d}\theta_o}{\mathrm{d}t} = 0$ is taken into account in Eq. (7) and notation $\lim_{t\to\infty} \theta_o(t) = \theta_{o_S}$, $\lim_{t\to\infty} \theta_L(t-T) = \theta_{L_S} = 1$ is used in what follows.Then Eq. (7) goes over on the form:

$$-K_1\theta_{o_S} + K_2\theta_{L_S} - \theta_{L_S} = 0\ , \qquad (31)$$

from where the stationary value is calculated

$$\theta_{o_S} = \frac{\theta_{L_S}(1-K_2)}{-K_1} = \frac{1\times(1-1)}{-1} = 0\ , \qquad (32)$$

which confirms expectation in the Fig.6 if the time t increases to$t\to\infty$.

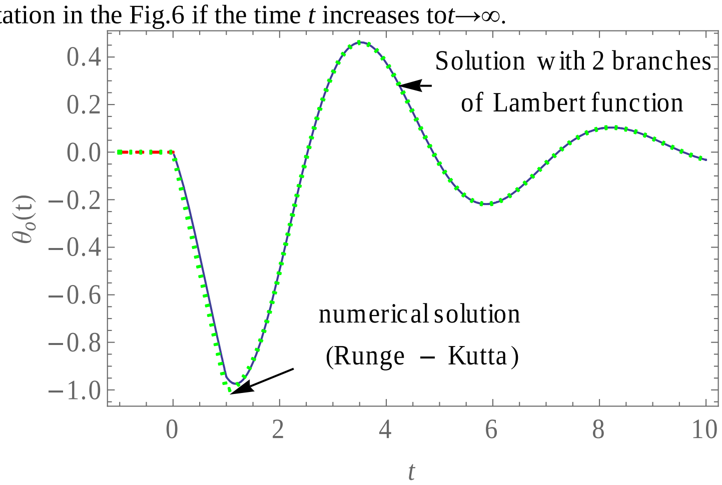

Figure 6. Comparison of the analytical solution of the control of production process with retained orders using 2 branches of Lambert function with the numerical solution using Runge-Kutta method. Computation for parameter values T=1, b_1=$-K_1$=−1, b_2=K_2=1 and N=0.

On the Fig. 7, the solution with 8 branches of Lambert function is shown. Note, that 8 branchescorresponds to the number N=3.By the help of Fig. 7 we can see, that solution computed by using Lambert function on the whole interval $[-T,10]$ almost perfectly matcheswith the numerical solution using Runge-Kutta method.

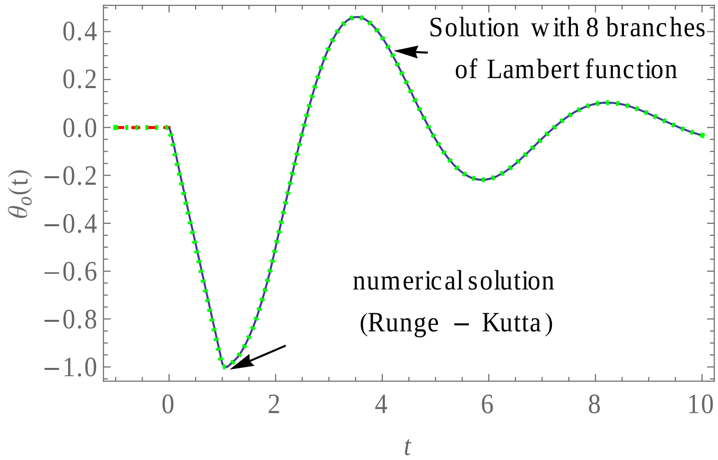

Figure 7. Comparison of the analytical solution of the control of production process with retained orders using 8 branches of Lambert function with the numerical solution using Runge-Kutta method. Computation for parameter values T=1, $b_1=-K_1=-1$, $b_2=K_2=1$ and N=3.

From the diagram in the Fig. 5 it followswith no doubt, that production system with the production delay T=1 and the rate K_1=1 of planned new production based on the information about control error $\varepsilon(t)$ is located in the stable area. With other words, it is located bellow the hyperbola, which indicates the stability boundof the branch $W_0(-K_1T)$. Consequently, the stable control of the production process results in the zero valued steady-state control error, $\lim_{t\to\infty} \varepsilon(t) = \lim_{t\to\infty}\left(\theta_{i_S} - \theta_{o_S}\right) = 0$. However, if the value of the constant K_1 is increased enough at the unchanged value of the time delay T, the production process can fall into unstable area. The unstable productionfor example happens (in accordance with the diagram in Fig. 5), if the constant K_1 at the unchanged value T=1 increases to K_1=2. When the production is unstable, this means practically, that the control of the production process doesn't work and oscillation amplitudes of actual inventory show an unlimited increase.Such a behaviour is produced by means of solution of Eq. (7) as is shown in the Fig. 8.

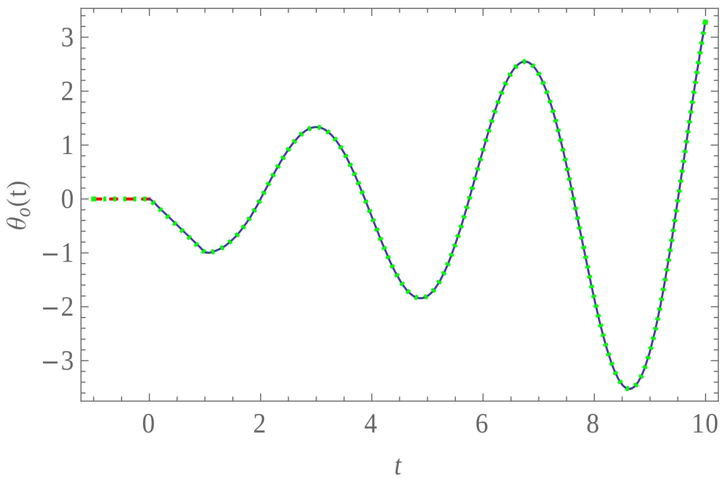

Figure 8. The comparison of the unstable analytical solution of production process with retained orders using 8 branches of Lambert function with the numerical solution using Runge-Kutta method. Computation for parameter values T=1, $b_1=-K_1=-2$, $b_2=K_2=2$ and N=3.

The comparison of results with numerical solution using Runge-Kutta method presents also in this case an almost perfect matching. The obtained result is surprising, because it proves, that the selected production delay causes an instability of the system, although orders do not oscillate at all, but are constant in the accordance with the assumption!

As the final example, let again the value K_1= 1 of the rate of planned new production based on the information about control error, but choose a smaller time delay T=0.3. The solution of Eq. (7) in this case is plotted in Fig. 9, where an aperiodic time response of actual inventory is obtained. The aperiodic behaviour is explained through an impact of the production delay producing an initial disturbance of the actual inventory level, which completely disappear as the time increases. The comparsion of obtained result by using 8 branches of Lambert function with numerical solution using Runge-Kutta method again shows an excellent agreement.

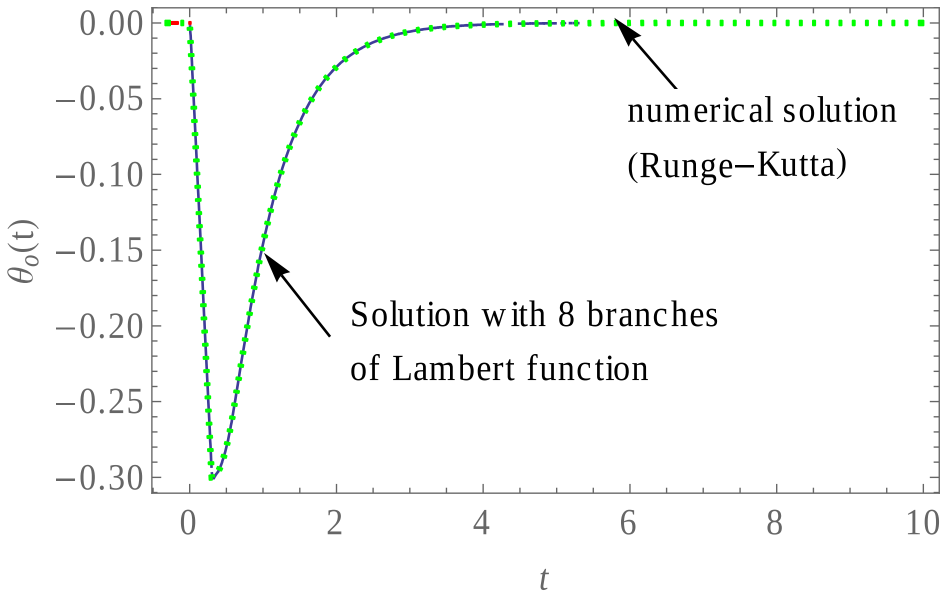

Figure 9. The comparison of aperiodic solution of production process with retained orders using 8 branches of Lambert function with the numerical solution using Runge-Kutta method. Computation for parameter values T=0.3, b_1=−K_1=−1, b_2=K_2=1 and N=3.

Remark. As we said in introduction, the delays are inherent property of every production process. Thus, the assumption that delay may be equal zero, T=0, is unrealistic and must be considered in theoretical sense as a limit case. Assuming the zero delay causes conversion of Eq. (7) into an ordinary differential equation. In addition,the prescribed function $\phi(t)$on the preshape interval is reduced into an initial condition $\theta_o(0)$ at the time instant t=0. Considering the initial condition $\theta_o(0) = 0$ and above selected values of parameters $K_1 = K_2$ =1, it is easy to show, that the initial disturbance in the time response of the actual inventory disappear. The actual inventory in such a case is equal zero, $\theta_o(t) \equiv 0$ for all times $t \geq 0$, as expected.

V. CONCLUSIONS

In this paper, an analytical method of solving the problem of the control of the production system with delay. For solving the governing linear differential equation with delay, Lambert functions are applied. The obtained solutions reveal, that production delay can cause an oscillatory or aperiodic time response of actual inventory in dependence on its extent. By means of Lambert function, an analytical stability bound of the production system is derived. By using the control of the production process, inventory oscillations are successfully damped, if the time delay and the rate of planned new production based on the information about the control error are located in the stable area. When both parameters lie in the unstable area, the control of the production system fails causing an unlimited increase of oscillation amplitudes of actual inventory regardless on the supposed constant loading of the system (the constant level of orders).

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