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On Invariance of ϵ- Orthogonality , ϵ- Approximation And ϵ- Coapproximation In Metric Linear Spaces

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ABSTRACT

We discuss ϵ — approximation , ϵ- coapproximation, ϵ- orthogonality, ϵ- approximation preserving, ϵ- coaaproximation preserving and ϵ-orthogonality preserving maps in metric linear spaces. The results proved in the paper generalize and extend several known results on the subject.

The Notion of othorgonality introduced by G.birkhoff [1] was used to characterize elements of best approximation in normed linear spaces (see [21], p.92). This notion of Orthogonality, extended to metric linear spaces was used to characterize elements of best approximation in [9] . A new kind of approximation , called best co-approximation was introduced and discussed in normed linear spaces by Franchetti and Furi [4] and subsequently many results on co-approximation appeared in normed linear spaces, metric linear spaces, metric spaces and other abstract spaces (see e.g.[10], [15]- [20] and reference cited therein). The notion of invariant best approximation in normed linear spaces was introducted and discussed by Meinardus [8] and thereafter Broswski [2] generalized result of Meinardus and proved some interesting results on invariance of best approximation. Various generalizations of their results appeared in literature since then in normed linear spaces (see e.g. [5]). Mazaheri and zadeh [7] discussed certain maps which preserve othrothogonality, best approximation and best co-approximation in normed linear spaces. The author in [12] extened the invariance principle of Meinardus to metric spaces and also discussed invariance of best approximation, best co-approximation and othorgonality in metric linear spaces in [14].

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I. INTRODUCTION

R.C. Buck [3] (see also [21] , p.162) introduced and discussed the notion of ϵ- approximation (called good approximation in [3]) in normed linear spaces and this notion was extended to metric spaces in [13] . Very little has been done so far concerning ϵ- approximation. This theory can be developed to a large extent parallel to the theory of best approximation. The notion of ϵ- co-approximation in metric spaces was introduced in [11]. It will be interesting to study elements of ϵ- co-approximation and develop a parallel theory, simillar to the theory of ϵ- approximation. Mazahari and Vaezpour [6] introduced the notion of ϵ-othrogonality and thereafter , Mazaheri and Zadeh [7] discussed ϵ- approximation preserving, ϵ- co-approximation preserving and ϵ- othrogonality preserving maps in normed linear spaces.

In this paper, we discuss ϵ- approximation, ϵ-co-approximation, ϵ- orthogonality, ϵ-approximation preserving, ϵ- co-approximation preserving and ϵ- othrogonality preserving maps in metric linear spaces. The proved results generalize and extend several results of [6], [7], [13] and [14].

To start with, we recall a few definitions.

Let G be a non-empty subset of a metric linear space (X,d) and $x \in X$. For a given $\varepsilon > 0$, an element $g_0 \in G$ is said to be an ϵ**-approximation** (ϵ**-coapproximation**) to x if d(x, g_0) $\leq$ d(x, g)+ ε for all $g \in G$ i.e. d(x,

g_0) $\le$ d(x, G)+ ε (d(g_0,g) $\le$ d(x, g)+ ε) for all $g \in G$. The set of all ϵ-approximation (ϵ-coapproximation) to x in G is denoted by $P_{G,\varepsilon}$ (x) ($R_{G,\varepsilon}$ (x)).

For $\varepsilon = 0$, we find elements of best approximation (best coapproximation) of x and respectively the sets P_G (x) (R_G (x)). The set G is said to be **ϵ-proximinal** (**ϵ-coproximinal**) if P_G (x) (R_G (x)) is non-empty for all $x \in X$.It is easy to see that elements of ϵ-approximation always exist but elements of ϵ-coapproximation may or may not exist.

For x,y $\in$ X, we say that x is ε **-orthogonal** to y, $x \perp_\varepsilon y$ if d(x,0) $\le$ d(x, α y)+ ϵ for all scalars α .For non-empty subsets A and B of X, we say that A is ε -orthogonal to B , $A \perp_\varepsilon B$, if $a \perp_\varepsilon b$ for all a $\in$ A , b $\in$ B. We define sets

$$\hat{G}_\varepsilon = \{ x \in X : x \perp_\varepsilon G \}$$

$$\check{G}_\varepsilon = \{ x \in X : G \perp_\varepsilon x \}$$

For a linear subspace G of a metric linear space (X,d) and $\varepsilon > 0$. we have

Proposition 1 $g_0 \in G$ is an ϵ-approximation to x $\in$ X if and only if $x - g_0 \in \hat{G}_\varepsilon$.

Proof $x- g_0 \in \hat{G}_\varepsilon \Leftrightarrow x- g_0 \perp_\varepsilon G$

$\Leftrightarrow$ $x- g_0 \perp_\varepsilon g$ for all g $\in$ G

$\Leftrightarrow$ d(x- g_0,0)$\le$ d(x- g_0 , α g) + ε for all g $\in$ G, for all scalars α

$\Leftrightarrow$ d(x , g_0)$\le$ d(x, $g_0 + \alpha$ g) + ε for all g $\in$ G, for all scalars α

$\Leftrightarrow$ d(x , g_0)$\le$ d(x, g')+ ε for all $g' \in$ G, for all scalars α

$\Leftrightarrow$ $g_0 \in P_{G,\varepsilon}$ (x)

Proposition 2 $g_0 \in P_{G,\varepsilon}(x) \Leftrightarrow 0 \in P_{G,\varepsilon}(x- g_0)$.

Proof $g_0 \in P_{G,\varepsilon}(x) \Leftrightarrow x- g_0 \in \hat{G}_\varepsilon$

$\Leftrightarrow$ $(x- g_0)-0 \in \hat{G}_\varepsilon$

$\Leftrightarrow$ $0 \in P_{G,\varepsilon}(x- g_0)$

Proposition 3 If $x- g_0 \in \check{G}_\varepsilon$ then $g_0 \in$ G is an ϵ-coapproximation to x .

Proof $(x- g_0) \in \check{G}_\varepsilon \Rightarrow G \perp_\varepsilon (x- g_0)$

$\Rightarrow$ $g \perp_\varepsilon (x- g_0)$ for all g $\in$ G

$\Rightarrow$ d(g,0)$\le$ d(g, α (x- g_0))+ ε for all g $\in$ G, for all scalars α

$\Rightarrow$ d(g+α g_0 ,α g_0)$\le$ d(g+α g_0 ,α x)+ ε for all g $\in$ G, for all α

$\Rightarrow$ d(g' ,α g_0)$\le$ d(g' ,α x)+ ε for all $g' \in$ G, for all α

$\Rightarrow$ d(g_0 , g')$\le$ d(x , g')+ ε for all $g' \in$ G

$\Rightarrow$ $g_0 \in R_{G,\varepsilon}$ (x)

$\Rightarrow$ $g_0 \in$ G is an ϵ-coapproximation to x

Proposition 4 Let G be a subspace of a metric linear space (X,d) and x $\in$ X. Then for $g_0 \in$ G , $G \perp_\varepsilon (x- g_0)$ $\Leftrightarrow$ α $g_0 \in R_{G,\varepsilon}$ (α x) for every scalar α .

Proof Let $\alpha\ g_0 \in R_{G,\varepsilon}(\alpha\ x)$ for all scalars α i.e. $d(\alpha\ g_0, g) \le d(\alpha\ x, g) + \varepsilon$ for all $g \in G$, for all scalars α .This implies $d(\alpha\ x - \alpha\ g_0, g - \alpha\ g_0) + \varepsilon \ge d(g - \alpha\ g_0, 0)$ for all $g \in G$, for all scalars α i.e. $d(g', \alpha\ (x - g_0)) + \varepsilon \ge d(g', 0)$ for all $g' \in G$ and all scalars α i.e. $G \perp_\varepsilon (x - g_0)$

Conversly,let $G \perp_\varepsilon (x - g_0)$ i.e. $g \perp_\varepsilon (x - g_0)$ for all $g \in G$. This implies $d(g, \alpha\ (x - g_0)) + \varepsilon \ge d(g,0)$ for all $g \in G$ and all scalars α .Therefore $d(g + \alpha\ g_0.\ \alpha\ x) + \varepsilon \ge d(g,0)$ for all $g \in G$ and all scalars α i.e. $d(g', \alpha\ g_0) \le d(g', \alpha\ x) + \varepsilon$ for all $g' \in G$ and all scalars α .

Therefore $\alpha\ g_0 \in R_{G,\varepsilon}(\alpha\ x)$ for all scalars α .

Proposition 5 $g_0 \in R_{G,\varepsilon}(x) \Leftrightarrow 0 \in R_{G,\varepsilon}(x - g_0)$

Proof $g_0 \in R_{G,\varepsilon}(x) \Leftrightarrow d(g_0, g) \le d(x, g) + \varepsilon$ for all $g \in G$

$\Leftrightarrow d(g_0 - g, 0) \le d(x - g_0, g - g_0) + \varepsilon$ for all $g \in G$

$\Leftrightarrow d(0, g - g_0) \le d(x - g_0, g - g_0) + \varepsilon$ for all $g \in G$

$\Leftrightarrow d(0.\ g') \le d(x - g_0, g') + \varepsilon$ for all $g' \in G$

$\Leftrightarrow 0 \in R_{G,\varepsilon}(x - g_0)$

For isometric mappings, we have

Theorem 1 let T be an isometery on a metric space (X,d) i.e. $d(Tx, Ty) = d(x, y)$ for all $x,y \in X$, $\varepsilon > 0$ and G be a subset of X such that T(G)= G. Then

(a) $T[R_{G,\varepsilon}(x)] \subseteq R_{G,\varepsilon}[Tx]$

(b) If x is T-invariant then $T[R_{G,\varepsilon}(x)] \subseteq R_{G,\varepsilon}(x)$

(c) If x is T-invariant and if $R_{G,\varepsilon}(x) = \{g_0\}$ then $Tg_0 = g_0$

(d) If x is T-invariant and if $\{g \in G : Tg = g\} \cap R_{G,\varepsilon}(x) = \varnothing$ then either $R_{G,\varepsilon}(x) = \varnothing$ or $R_{G,\varepsilon}(x)$ has more than one point.

Proof (a) Let $T(g_0) \in T[R_{G,\varepsilon}(x)]$ i.e. $g_0 \in R_{G,\varepsilon}(x)$.

Let $g \in G$ be arbitrary. Then T(G)= G implies the existence of $u \in G$ such that g=T(u). Consider

$$\begin{aligned} d(Tg_0, g) = d(Tg_0, Tu) = d(g_0, u) &\le d(x, u) + \varepsilon \\ &= d(Tx, Tu) + \varepsilon \\ &= d(Tx, g) + \varepsilon \text{ for all } g \in G \end{aligned}$$

This implies that $T(g_0) \in T[R_{G,\varepsilon}(x)]$whenever g_0 is an ε -coapproximation to x.

(b) Suppose $g_0 \in R_{G,\varepsilon}(x)$. Then (a) implies $T(g_0) \in R_{G,\varepsilon}[Tx]$ i.e. $T(g_0) \in R_{G,\varepsilon}(x)$ i.e. $T[R_{G,\varepsilon}(x)] \subseteq R_{G,\varepsilon}(x)$

(c) By (b), $T(g_0) \in \{g_0\}$i.e. $T(g_0) = g_0$

(d) By (b), $T(g_0) \in R_{G,\varepsilon}(x)$. But by the hypothesis , no invariant element can be an ε -coapproximation, therefore $T(g_0) \ne g_0$. So, if $T(g_0) = g_0$ then an ε -coapproximation to x does not exist i.e. $R_{G,\varepsilon}(x) = \varnothing$, If $T(g_0) \ne g_0$ then x has at least two ε -coapproximations to x.

Remarks It is easy to see that similar results are true for $P_{G,\varepsilon}(x)$.

The next result will be useful in our subsequent discussion:

Lemma Let (X,d) be a metric linear space .If $T: X \to X$ is an isometry then for all subspaces G of X and $x \in X$, $T[P_{G,\varepsilon}(x)] = P_{T(G),\varepsilon}[Tx]$ and $T[R_{G,\varepsilon}(x)] = R_{T(G),\varepsilon}[Tx]$

Proof Since T is an isometry , d(Tx,Ty) = d(x,y) for all $x,y \in X$. The proof now follows from d(x , g_0) ≤ d(x, g) + ε for all g ∈ G ⇔ d(Tx,T g_0) ≤ d(Tx, Tg) for all Tg ∈ T(G) and d(g_0 ,g) ≤ d(x, y) + ε for all g ∈ G ⇔ d(T g_0 ,Tg) ≤ d(Tx, +Tg) + ε for all T(g) ∈ T(G).

Definition Suppose X and Y are metric linear spaces and $\varepsilon > 0$. A map T:X → Y is called ϵ-**approximation preserving** (ϵ- **coaaproximation preserving**) if for all subspaces G of X and all x ∈ X, T[$P_{G,\varepsilon}$ (x)]= $P_{T(G),\varepsilon}$ [Tx] (T[$R_{G,\varepsilon}$ (x)]= $R_{T(G),\varepsilon}$ [Tx])

Above lemma shows that if (X,d) is a metric linear space then every isometry T:X → X is ϵ-approximation (ϵ- coaaproximation) preserving .As a consequence of the above lemma , we obtain

Theorem 2 Suppose (X , d) and (Y, d') are two metric linear spaces and T:X → Y is a linear map which is an isometry . Then

(a) A subspace G of X is ϵ-proximinal (ϵ-coproximinal) if and only if T(G) is ϵ-proximinal (ϵ-coproximinal)

(b) A subspace G of X is ϵ-Chebyshev (ϵ-coChebyshev) if and only if T(G) is ϵ- Chebyshev (ϵ-coChebyshev)

Theorem 3 Suppose X and Y are metric linear spaces, $\varepsilon > 0$ and T: X → Y is a linear onto isometry . Then

(a) $x \perp_\varepsilon y \Leftrightarrow Tx \perp_\varepsilon Ty$

(b) For a subspace G of X, $T(\hat{G}_\varepsilon) = \widehat{T(G)}_\varepsilon$

(c) For a subspace G of X, $T(\check{G}_\varepsilon) = \widecheck{T(G)}_\varepsilon$

Proof

(a) $x \perp_\varepsilon y \Leftrightarrow$ d(x, 0) ≤ d(x , α y) + ε for all scalars α

⇔ d(Tx, T0) ≤ d(Tx , T(α y)) + ε for all scalars α

⇔ d(Tx, T0) ≤ d(Tx , α T(y)) + ε for all scalars α

⇔ $Tx \perp_\varepsilon Ty$

(b) Let y ∈ $T(\hat{G}_\varepsilon)$. Then y = Tx , x ∈ $\hat{G}_\varepsilon$. Now x ∈ $\hat{G}_\varepsilon$ ⇒ $x \perp_\varepsilon G$ ⇒ $Tx \perp_\varepsilon T(G)$ ⇒ $y \perp_\varepsilon T(G)$ ⇒ y ∈ $\widehat{T(G)}_\varepsilon$. Therefore $T(\hat{G}_\varepsilon) \subseteq \widehat{T(G)}_\varepsilon$. Conversely , suppose y ∈ $\widehat{T(G)}_\varepsilon$. Then $y \perp_\varepsilon T(G)$. Since T is onto ,y = Tx , x ∈ X and so Tx $\perp_\varepsilon$ T(G) i.e Tx $\perp_\varepsilon$ Tg for all g ∈ G. Therefore d(Tx, 0) ≤ d(Tx , α T(g)) + ε for all g ∈ G and all scalars α i.e. d(Tx, 0) ≤ d(Tx , T(α g)) + ε for all g ∈ G and all scalars α .Therefore d(x, 0) ≤ d(x , α g) + ε for all g ∈ G and all scalars α i.e. $x \perp_\varepsilon g$ for all g ∈ G i.e $x \perp_\varepsilon G$ and so x ∈ $\hat{G}_\varepsilon$ i.e. y = Tx ∈ $T(\hat{G}_\varepsilon)$. Therefore $\widehat{T(G)}_\varepsilon \subseteq T(\hat{G}_\varepsilon)$ and hence $T(\hat{G}_\varepsilon) = \widehat{T(G)}_\varepsilon$.

(c) Let y ∈ $T(\check{G}_\varepsilon)$. Then y=Tx, x ∈ $\check{G}_\varepsilon$. Now x ∈ $\check{G}_\varepsilon$ ⇒ $G \perp_\varepsilon x$ ⇒ $T(G) \perp_\varepsilon Tx$ ⇒ $T(G) \perp_\varepsilon y$ ⇒ y ∈ $\widecheck{T(G)}_\varepsilon$. Conversly , suppose y ∈ $\widecheck{T(G)}_\varepsilon$ i.e T(G) $\perp_\varepsilon$ y . Since T is onto ,y = Tx , x ∈ X and so T(G) $\perp_\varepsilon$ Tx i.e Tg $\perp_\varepsilon$ Tx for all g ∈ G and therefore d(Tg,0) ≤ d(Tg , T(α x)) + ε for all g ∈ G and all scalars α . Therefore d(g, 0) ≤ d(g , α x) + ε for all g ∈ G and all scalars α i.e $G \perp_\varepsilon x$ and so x ∈ $\check{G}_\varepsilon$ i.e. y = Tx ∈ $T(\check{G}_\varepsilon)$. Therefore $\widecheck{T(G)}_\varepsilon \subseteq T(\check{G}_\varepsilon)$ and hence $T(\check{G}_\varepsilon) = \widecheck{T(G)}_\varepsilon$.

Theorem 4 If G is a linear subspace of a metric linear space (X,d) and $\varepsilon > 0$, then

(a) G is ϵ-proximinal ⇔ $X = G + \hat{G}_\varepsilon$

(b) G is ϵ-Chebyshev $\Leftrightarrow$ $X = G \oplus \hat{G}_\varepsilon$

(c) G is ϵ-semi Chebyshev $\Leftrightarrow$ each x $\in$ X has atmost one sum decomposition as $G + \hat{G}_\varepsilon$

Proof (a) Suppose G is ϵ-proximinal and x $\in$ X is arbitrary . Since G is ϵ-proximinal , there exists $g_0 \in P_{G,\varepsilon}$ (x) and so x- $g_0 \in \hat{G}_\varepsilon$ and x= g_0 + (x- g_0) $\in$ $G + \hat{G}_\varepsilon$ .Hence $X = G + \hat{G}_\varepsilon$.

Conversly , suppose $X = G + \hat{G}_\varepsilon$. Let x $\in$ X be arbitrary. Then x= g_0 + (x- g_0). Now x- $g_0 \in \hat{G}_\varepsilon \Rightarrow g_0 \in P_{G,\varepsilon}$ (x) and hence G is ϵ-proximinal in X.

Let G be ϵ-Chebyshev in X. Then G is ϵ-proximinal in X and so by (a) , $X = G + \hat{G}_\varepsilon$. Let x $\in$ X be such that x= $g_1 + y_1 = g_2 + y_2$; $g_1 \in$ G, $g_2 \in$ G, $y_1 \in \hat{G}_\varepsilon$, $y_2 \in \hat{G}_\varepsilon$ .This gives $g_1 - g_2 = y_2 - y_1 \in$ G. Now $y_1 \in \hat{G}_\varepsilon$ $\Rightarrow$ $y_1 - 0 \in \hat{G}_\varepsilon \Rightarrow 0 \in P_{G,\varepsilon}(y_1) \Rightarrow g_1 \in P_{G,\varepsilon}(y_1 + g_1)$ i.e. $g_1 \in P_{G,\varepsilon}$ (x).Similarly , $g_2 \in P_{G,\varepsilon}$ (x). Since G is ϵ-Chebyshev, $g_1 = g_2$ and so $y_2 = y_1$ i.e. x $\in$ X has a unique representation and hence $X = G \oplus \hat{G}_\varepsilon$

Conversly , suppose $X = G \oplus \hat{G}_\varepsilon$. To show that G is ϵ-Chebyshev. Since X= $G + \hat{G}_\varepsilon$, G is ϵ-proximinal by (a).Suppose x $\in$ X has two distinct ϵ-approximations in G, say g_1 and g_2 . Then x- g_1 , x- $g_2 \in \hat{G}_\varepsilon$ .But then x= g_1 + (x- g_1) and x= g_2 + (x- g_2). This contradicts $X = G \oplus \hat{G}_\varepsilon$.

(c) follows from proofs of (a) and (b).

Remarks Can we prove similar results for co- approximation ?We know that ϵ- approximation always exists but ϵ- coapproximation may or may not exist. However, we have

Theorem 5_If G is a linear subspace of a metric linear space (X,d) such that X= $G + \check{G}_\varepsilon$ then G is ϵ-coproximinal in X.

Proof Let x $\in$ X be arbitrary. Then x= g_0 +y $\in$ $G + \check{G}_\varepsilon$. Now y = y-0 $\in \check{G}_\varepsilon \Rightarrow 0 \in R_{G,\varepsilon}$ (y) i.e. $0 \in R_{G,\varepsilon}$ (x- g_0) and so $g_0 \in R_{G,\varepsilon}$ (x). Hence G is ϵ-coproximinal in X.

REFERENCES

[1] G. Birkhoff, Orthogonality in linear metric spaces, Duke Math. J. (1935), 169-172.
[2] Bruno Brosowski, Fixpunktsatze in der approximations theorie, Mathematics (Cluj), 11 (1969), 195-220.
[3] R.C. Buck, Applications of duality in approximation theory (Ed.H.L. Barabedian), Elsevier, Amsterdam-London-New York (1965), 27-42.
[4] C. Franchetti and M.Furi, Some characteristic properties of real Hilbert spaces, Rev. Roum. Math Pures Appl. 17 (1972), 1045-1048.
[5] L.A. Khan and A.R. Khan, An extension of Brosowski-Meinardus theorem on invariant approximation, Approx. Theory & Appl. 11 (1995), 1-5.
[6] H. Mazaheri and S.M. Vaezur, Orthogonality and e-orthogonality in Banach spaces, The Australian Journal of Mathematical Analysis and Applications 2 (2005), 1-5.
[7] H. Mazaheri and M. Hossein Zadeh, The maps preserving approximation, 2 (2007), 905-909
[8] G. Meinardus, Invariance bei linearen approximation, Arch. Rational Mech. Anal. 14 (1963), 301-303.
[9] T.D. Narang, On certain characterizations of best approximation in metric linear spaces, Pure Appl. Mathematica Sciences, 4(1976), 121-124.
[10] T.D. Narnag, On best coapproximation in normed linear spaces, Rocky Mountain Journal of Mathematics 22 (1991),

[11] T.D.Narang, On good approximation, Journal of Scientific Research, 49 (1999), 25-43.
[12] T.D.Narang, Invariant approximation in metric spaces, Mathematics Today, 23 (2007),19-24.
[13] T.D.Narang, On simultaneous characterization of set of elements of good approximation in metric spaces, Affrican Diaspora Journal of Mathematics, 10 (2012), 1-8
[14] T.D.Narang, On invariance of approximation and coapproximation in metric linear spaces, communicated
[15] T.D.Narang and S.P. Singh, Best coapproximation in metric linear spaces, Tamkang Journal of Mathematics, 30 (1999), 241-252
[16] P.L. Papini and I. Singer, Best approximation in normed linear spaces, Math. Mh. 88 (1979), 27-44.
[17] Geetha S. Rao, Tools and techniques in approximation theory, The Mathematics Student, 81 (2012), 115-133.
[18] Geetha S. Rao, A new Tool in approximation theory, The Mathematics Student, 83 (2014), 5-20.
[19] Geetha S. Rao and R. Sarvanan, Some results concerning uniform best coapproximation, J. Inequel. Pure Appl. Math 3 (2002), 1-13.
[20] Greth. S Rao and R. Sarvan, Strongly unique best coapproximation, Kyungpook Math. J. 43 (2003), 519-538
[21] I. Singer, Best approximation in Normed Linear Spaces by Elements of Linear Subspaces, Springer-Verlag, New York, 1970.