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Numerical Solution Of Delay Differential Equations Using The Adomian Decomposition Method(ADM)

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ABSTRACT
Adomian Decomposition Method has been applied to obtain approximate solution to a wide class of ordinary and partial differential equation problems arising from Physics, Chemistry, Biology and Engineering. In this paper, a numerical solution of delay differential Equations (DDE) based on the Adomian Decomposition Method (ADM) is presented. The solutions obtained were without discretization nor linearization. Example problems were solved for demonstration.
Keywords: *Adomian Decomposition, Delay Differential Equations (DDE), Functional Equations , Method of Characteristic.*

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I. INTRODUCTION

Problems arising from modelling real life problems usually end up with differential, partial differential or delay differential equations, etc. Solution of such problems by the usual procedures of mathematical analysis such as linearization and perturbation techniques, assumptions of weak nonlinearity etc, necessarily have to make slight changes to such problems in order to make them mathematically solvable. Thus, solutions derived from such procedures (i.e. linearization, perturbation etc) may not actually reflect the actual behaviour of the original problem thereby putting the use of such results to serious limitations. The avoidance of these deviations so that physically correct solutions can be obtained is highly desirable. This is the sole objective of the Adomiam Decomposition Method (ADM) [1]

ADM seeks to make possible physically realistic solutions of complex real life problems without the usual modeling and mathematical compromises to achieve results. It has been shown [3] that nonlinear systems can be extremely sensitive to small changes, thus the ADM is the most desirable option when solving such problems.
Delay Differential Equations (DDEs) generally provide the best and sometimes the only realistic simulation of observed phenomena. [Baker and Paul (1992)] and [Zennaro(1988)].

DDE of the first order has the form: $y^{/}(t) = f(t, y(t), y(g(t)),\quad t \geq 0,$

$$y(t) = \emptyset(t), \qquad t < t_0 \qquad (1)$$

or equivalently as: $y^{'}(t) = f(t, y(t), y(t-\tau_1), y(t-\tau_2), \ldots . y(t-\tau_n))$
where the delays or lags τ_i are always nonnegative, may be positive constants, $0 < \tau_1 < \tau_2 < \cdots . \tau_k$ or functions of t , $\tau_i = \tau_i$ (t) or functions of t and y itself, $\tau_i = \tau_i(t, y(t))$. Thus essentially DDE types are classified according to the complexity of the phenomenon namely
(i) DDE with constant delay, (ii) DDE with variable or time dependent delay and (iii) DDE with state dependent delay.
Though Delay Differential Equations (DDE) arise in models through the sciences, DDE with constant delays is a large class, especially popular for Biological models [2]. Models in Immunology, Spread of Infection, Predator-Prey etc are common instances in which DDEs occur frequently.
There are lots of differences in finding the solution of ODE initial value problem and a Delay Differential equation. For instance given a DDE :

$$y^{/}(t) = f(t, y(t), y(t-T)); \quad t \geq t_0 \qquad (2)$$
$$y(t) = \varphi(t), \quad t \leq t_0$$

and ODE IVP: $y^{/}(t) = f(t, y)$, $y(a) = c$, $t \geq a$ (3)
The most obvious difference in solving ODE initial value problem(3) and DDE (2) is that for some $t \geq t_0$, it can be that $t - T < t_0$; thus solution of (2) is determined by value at initial point and values prior to the initial point, which is called the history . The given initial data must include $y(t_0)$, the initial value and values y(t), for $t \leq a$, say for all t in [a- T, a].

Adding a small delay term in an ODE can change the qualitative behaviour of solutions of the equations. [2],[3]. For instance, in the equation

$y^{/}(t) = y^2(t)$, $y(0) = 1$, $t \geq 0$

the solution is not defined for finite points $t \geq t^* > 0$. But the corresponding DDE

$y^{/}(t) = y(t)y(t-1)$, $y(t) = 1$ $t \leq 0$ has solutions for all $t \geq 0$.

Likewise , all the solutions of the ODE: $y^{/}(t) = -y(t)$ are all decaying exponentials, whereas the solution of the DDE $y^{/}(t) = -y(t-\pi/2)$ has solutions of the form $y(t) = A\sin(t) + B\cos(t)$.

For other details of the theories and other qualitative properties of solutions of DDE see [3].

The analytic methods for solving Delay Differential Equation include: (i)The Method of Characteristics,(ii)The Bellman's Method of Steps and (iii) The Method of Laplace.

Until recently the most common numerical method for solving DDE is the continuous extension of Runge Kutta methods and its variants [3] . The limitations of these methods have been highlighted earlier in the introduction.

In this paper our focus is on the use of the Adomian Decomposition Method to solve different types of Delay Differential Equations. In the next section we present the essential highlights of the method and then go further to the implemention.

II. THE THEORY OF THE METHOD

Given the deterministic form $Fu = g(t)$, where F is a nonlinear ODE operator with linear and nonlinear terms . This can be written as $Lu + Ru + Nu = g$

Here L is the operator for the highest ordered derivative, R is the remainder of the linear term and N is the nonlinear term.

Thus we have $Lu = g - Ru - Nu$

Applying the inverse operator L^{-1} (the n – fold definite integration operator from 0 to t)

$\Longrightarrow L^{-1}Lu = L^{-1}g - L^{-1}Ru - L^{-1}Nu$

For example if $L = \frac{d^2}{dt^2}$ then $L^{-1}Lu = u - u(0) - tu^{/}(0)$

$\Longrightarrow u = u(0) + tu^{/}(0) + L^{-1}g - L^{-1}Ru - L^{-1}Nu$

The method show that the solution u(t) can be written as a series thus

$u(t) = \sum_{n=0}^{\infty} u_n = u_0 - L^{-1} R \sum_{n=0}^{\infty} u_n - L^{-1} \sum_{n=0}^{\infty} A_n$ (4)

In the simple case where there is no nonlinear term then the solution u(t) is readily obtained by

$u(t) = \sum_{n=0}^{\infty} u_n$ where the components u_n are determined recursively (5)

In the case where there is nonlinear term Nu , this is defined by the Adomian polynomials

$Nu = \sum_0^{\infty} A_n$, where A_n are the Adomian Polynomial that can be generated for all forms.

In [1] there are defined formulas for generating the polynomials. The first four are reproduced below.

$$A_0 = f(u_0) \qquad A_1 = u_1 f^{(1)}(u_0) \qquad A_2 = u_2 f^{(1)}(u_0) + \frac{1}{2!}u_1^2 \ f^{(2)}(u_0)$$

$$A_3 = u_3 f^{(1)}(u_0) + u_1 u_1 \frac{1}{2!}u_1^2 \ f^{(2)}(u_0) + \frac{1}{3!}u_1^3 \ f^{(3)}(u_0)$$

$$A_4 = u_4 f^{(1)}(u_0) + [u^2{}_2 \frac{1}{2!} + u_1 u_3] \ f^{(2)}(u_0) + u_1^2 u_2 \frac{1}{2!} \ f^{(3)}(u_0) + u_1^4 \frac{1}{4!} f^{(4)}(u_0)$$

The A_i s depend only on the components u_0 to u_i

It has been shown that the series terms approach zero as $1/_{mn!}$ if m is the order of the highest linear differential operator [1].

Another interesting property is that since the series converges rapidly in norm then the partial sum $\emptyset_n = \sum_{i=0}^{n-1} u_i$ can serve as guide for practical design and $\lim_{n\to\infty} \emptyset_n = u$

III. EXAMPLE PROBLEMS

3.1 solve $y^{/}(x) = 1 - 2y^2\left(\frac{x}{2}\right)$, $0 \leq x \leq 1$, $y(0) = 0$, $x \leq 0$

$Ly = 1 - 2y^2 \left(\frac{x}{2}\right)$, $\Longrightarrow y(x) = L^{-1}\left(1 - 2y^2 \left(\frac{x}{2}\right)\right) = x - L^{-1}(2y^2(\frac{x}{2}))$

$y_0(x) = y(0) = x$ and $y_{n+1} = -2\int_0^x A_n dx$ $n \geq 0$ (6)

The nonlinear term is $f(y) = y^2(x/2)$. The first Adomian polynomial $A_0 = f(y_0)$, then using the generating formula in section 2 , we can compute the remaining A_n s.

$A_o = y_0^2\left(\frac{x}{2}\right)$

$A_1 = 2y_0\left(\frac{x}{2}\right)\, y_1\left(\frac{x}{2}\right)$, $\quad A_2 = 2y_2\left(\frac{x}{2}\right)\, y_0\left(\frac{x}{2}\right) + y_1^2\left(\frac{x}{2}\right)$

$A_3 = 2y_3\left(\frac{x}{2}\right)\, y_0\left(\frac{x}{2}\right) + 2y_1\left(\frac{x}{2}\right)\, y_2\left(\frac{x}{2}\right) \quad \ldots\ etc$

Now using (6) we compute the components y_{n+1}

$y_1 = -2\int_0^x \frac{x^2}{4}\,dx = -\frac{x^3}{6}, \qquad y_1\left(\frac{x}{2}\right) = -\frac{x^3}{48}, \qquad y_2 = \int_0^x \frac{1}{24}x^4 dx = -\frac{x^5}{6}, \quad y_2\left(\frac{x}{2}\right) = \frac{x^5}{3840}$

$y_3 = \int_0^x \frac{-2}{2304}x^6 - \frac{2x^6}{3840}dx = -\frac{x^7}{5040}, \qquad y_3\left(\frac{x}{2}\right) = -\frac{x^7}{645120}$

Now the partial sum φ_3 *is*

$y_0 + y_1 + y_2 + y_3 = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7$ (7)

$y_4 = \int_0^x \frac{4x^3}{48}.\frac{x^5}{3840}2x.\frac{1}{645120}x^7 dx = \frac{x^9}{362880}$

Now the partial sum $\varphi_4 = \sum_{i=1}^4 y_i = x - \frac{x^3}{6} + \frac{x^5}{120} + \frac{x^7}{5040} + \frac{x^9}{362880}$ (8)

Using (8) to compute approximate y(x) values are as shown in table 3.1. These compare favourably with the exact solution y(x) = sinx

Table 3.1

X	ADM	Exact	Error
0.2	0.1986693309	0.1986693308	-1E-10
0.4	0.3894183422	0.3894183423	1E-10
0.6	0.5645424735	0.5646424734	9.99999E-05
0.8	0.717356093	0.717356090	-3E-09

3.2 Solve $y^{/}(x) = \frac{1}{2}e^{\frac{x}{2}}y\left(\frac{x}{2}\right) + \frac{1}{2}y(x); \qquad 0 \le x \le 1, \qquad y(0) = 1$

$y_0 = 1, \quad y_{n+1} = \int_0^x \frac{1}{2}e^{\frac{x}{2}}y_n\left(\frac{x}{2}\right) + \frac{1}{2}y_n(x)dx, \quad n \ge 0$ (9)

$y_1 = \int_0^x \frac{1}{2}e^{\frac{x}{2}} + \left(\frac{1}{2}\right)dx = -1 + e^{\frac{x}{2}} + \frac{x}{2}$

$y_2 = -\frac{1}{6} + \frac{2}{3}e^{\frac{3}{4}x} - \frac{1}{2}e^{\frac{1}{2}x} + \frac{1}{4}xe^{\frac{1}{2}x} - \frac{1}{2}x + \frac{1}{8}x^2$

$y_3 = \frac{151}{36} + \frac{5}{12}e^{\frac{1}{2}x} + \frac{16}{9}e^{\frac{3}{16}x} - 6e^{\frac{1}{8}x} + \frac{1}{2}xe^{\frac{1}{8}x} - \frac{1}{8}x\,e^{\frac{1}{2}x} + \frac{1}{32}x^2e^{\frac{1}{2}x} - \frac{1}{12x} + \frac{4}{9}e^{\frac{3}{4}x} - \frac{1}{8}x^2 + \frac{1}{48}x^3$

$y_4 = \frac{49787}{216} + \frac{287}{72}e^{\frac{1}{2}x} - \frac{107}{3}e^{\frac{1}{8}x} + \frac{512}{27}e^{\frac{3}{64}x} - 224e^{\frac{1}{32}x} + 4x\,e^{\frac{1}{32}x} + \frac{5}{4}xe^{\frac{1}{8}x} + \frac{1}{32}x^2e^{\frac{1}{8}x} - \frac{5}{48}xe^{\frac{1}{2}x} + \frac{160}{72}e^{\frac{3}{16}x} - \frac{1}{64}x^2e^{\frac{1}{2}x} + \frac{1}{384}x^3e^{\frac{1}{2}x} + \frac{151}{72}x$

This computation continued until y_{12} term before obtaining convergence to the exact solution.

3.3 Solve $\frac{d^2y}{dx^2} = \frac{3}{4}y(x) + y\left(\frac{x}{2}\right) - x^2 - 2 \quad \frac{d^2y}{dx^2} = \frac{3}{4}y\,(x) + y\left(\frac{x}{2}\right) - x^2 - 2, \quad 0 \le x \le 1$, $\quad y(0) = 0$, $y^{-}(0) = 0$, $x \le 0$

$y_0(x) = \iint_0^x \frac{3}{4}(x) + y(x) - x^2 + 2\,dxdx$ evaluated at x=0 gives

$y_0(x) = \iint_0^x \frac{3}{4}y(x) + y(x) - x^2 + 2dxdx \iint_0^x \frac{3}{4}y(0) + y\,(0) - x^2 + 2dxdx$

$\Rightarrow y_0(x) = x^2 - \frac{x^4}{12}$, and $y_{n+1}(x) = \iint_0^x \frac{3}{4}y_n(x) + y_n\,(x) - x^2 + 2dxdx\; n > 0$ (10)

$$y_1 = \int_0^x \int_0^x \frac{3}{4}\left(x^2 - \frac{1}{12}x^4\right) + \frac{x^2}{4} - \frac{x^2}{192} dx\, dx = \frac{191}{2304}x^4 - \frac{1}{480}x^6$$

$$y_2 = \int_0^x \int_0^x \frac{3}{4}\left(\frac{191}{2304}x^4 - \frac{1}{480}x^6\right) + \frac{191}{36864}x^4 - \frac{1}{30720}x^6 dx\, dx = \frac{2483}{1105920}x^6 - \frac{7}{245760}x^8$$

$$y_3 = \int_0^x \int_0^x \frac{3}{4}\left(\frac{2483}{1105920}x^6 - \frac{7}{245760}x^8\right) + \frac{2483}{70778880}x^6 - \frac{7}{62914560}x^8 dx\, dx$$

$$= \frac{17381}{566231040}x^8 - \frac{1351}{5662310400}x^{10}$$

$$y_4 = \frac{3354533}{13045963161600}x^{10} - \frac{1038919}{765363172147200}x^{12}$$

$$y_5 = \frac{2579635877}{1763396748627148800}x^{12} - \frac{456085441}{81508116380988211200}x^{14}$$

$$y_6 = \frac{198433529}{57782984659014411878400}x^{14} - \frac{3192598087}{128201196346664182087680000}x^{16}$$

$$y_7 = \frac{2438549637881}{227211940956790109811769344000} - \frac{12071213366947}{1977652734087901115801640370176000}x^{18}$$

$y(x) = \sum_{t=0}^{7} y_i$ gives the partial sum approximation of the solution values. The comparison with exact values is given in table 3.2

Table 3.2

X	y(x) ADM	y(x) exact solution
0.0	0.0	0.0
0.2	0.03999993153	0.04
0.4	0.1599895534	0.16
0.6	0.3599513389	0.36
0.8	0.6398650251	0.64
1.0	0.9997300599	1.0

3.4 Solve $\frac{dy}{dx} = [1 + y(x)]y\left(\frac{x}{2}\right)$, $1 \le x \le 4$, $y(x) = 1$, $0 \le x \le 1$

$y_0 = \int_1^x (1 + y(x))y\left(\frac{x}{2}\right) dx = \int_1^x 2dx = 2x - 2,$

$y_0\left(\frac{x}{2}\right) = x - 2,$ $\qquad y_{n+1} = \int_1^x [1 + y_n(x)]y\left(\frac{x}{2}\right) dx$ (11)

$$y_1 = 2x - \frac{1}{6} + \frac{2}{3}x^3 - \frac{5}{2}x^2$$

$$y_2 = -\frac{5}{36}x - \frac{359}{2016} + \frac{1}{126}x^7 - \frac{5}{48}x^6 + \frac{23}{48}x^5 - \frac{91}{96}x^4 + \frac{91}{144}x^3 + \frac{1}{4}x^2$$

$$y_3 = -\frac{594863}{4064256}x + \frac{4789}{870912}x^3 - \frac{1565}{96768}x^2 - \frac{785543}{130056192}x^7 + \frac{62221}{6193152}x^6$$
$$+ \frac{107329}{6635520}x^5 - \frac{905}{49152}x^4 + \frac{52760122091}{334764638208} - \frac{393037}{43352064}x^8 + \frac{891773}{83607552}x^9$$

$$-\frac{883}{172032}x^{14}+\frac{1}{30481920}x^{15}+\frac{985}{40255488}x^{13}-\frac{47}{196608}x^{12}+\frac{41287}{29196288}x^{11}$$

$$y_4 = \frac{2044585366660749665209}{1120673629945331334 51264}x-\frac{361507333262608391}{32653660546192637 9520}x^9$$

$$+\frac{1724824761919}{340142297356173312}x^3-\frac{130954592996581285}{1360569189424693248}x^2$$

$$+\frac{1415686669370697601}{108845535153975459840}x^7+\frac{5883282668242351}{1332802471273168896}x^6$$

$$-\frac{13250164149886973}{259156036080893952 0}x^5+\frac{1744394411483495}{58101081182011392}x^4$$

$$-\frac{22977368275584935}{1583207784057824870 4}x^8+\frac{1605814479948579829}{97960981638577913 8560}x^{10}$$

$$-\frac{618262972935793}{10719636037891522560}x^{14}+\frac{3984471945937759}{148425729755421081600}x^{15}$$

$$-\frac{2740702697436551}{17326432126551195 6480}x^{13}+\frac{84668516381708137}{293216543680097157120}x^{12}$$

$$-\frac{189594690223731469}{213803729766737510400}x^{11}-\frac{1651725393550334317615063647444 1}{18711526181951539923002169753600 0}$$

$$+\frac{2396541839181487}{373184691956487290880}x^{16}+\frac{1}{28803570853478400}x^{31}$$

$$-\frac{1}{330363536670720}x^{30}+\frac{48389}{398550570639556608}x^{29}$$

$$-\frac{90983}{30540273612226560}x^{28}+\frac{1628218573}{324870602372834918 40}x^{27}$$

$$-\frac{18900269}{30809588194344960}x^{26}+\frac{26357624082151}{466480864945609113 6000}x^{25}$$

$$-\frac{11578815968243}{290254760410601226240}x^{24}+\frac{1995670980206023}{917930679798526377 9840}x^{23}$$

$$-\frac{121196867155951}{133033431854858895360}x^{22}+\frac{4705550936141747}{1632683027309631897600}x^{21}$$

$$-\frac{103272035359}{15670810949713920}x^{20}-\frac{337671295473833}{345317003920775577 60}x^{19}$$

$$-\frac{144654049387121}{23324043247280556 80}x^{18}-\frac{10333140079326283}{1850374097617582817280}x^{17}$$

With the aid of Maple software $y_{5,}$ y_6 , y_7, were calculated . The approximator y(x) = $\sum_{i=0}^{7} y_i$ gives approximated solutions which converges to the exact solution.

III. CONCLUSION

In this paper, a numerical method based on the Adomian decomposition method (ADM) was implemented to solve both linear and nonlinear delay differential equations. The numerical results obtained by using the Adomian decomposition method compares favourably with the exact solutions. The attractive property of the method is that it is implemented directly in a straightforward manner without using restrictive assumptions, linearization, and discretization. The efficiency of the decomposition method gives it much wider applicability which has to be explored further.

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