The International Journal Of Engineering And Science (IJES)
|| Volume || 4 || Issue || 4 || Pages || PP.49-54|| 2015 ||
ISSN (e): 2319 – 1813 ISSN (p): 2319 – 1805

# On the Construction of Cantor like Sets

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**ABSTRACT**

*In this paper we construct a Cantor like set $S$ from any sequence $\{ \epsilon_n \}$ with $0< \epsilon_n <1$ with the help of sequence $\{ S_n \}$ of subsets of [0,1] such that $S_n \supset S_{n+1}$, $m(S_n) = \prod_{j=1}^{n}(1-\epsilon_j)$ and $S = \cap S_n$ with $m(S) = \prod_{n=1}^{\infty}(1-\epsilon_n)$ .Further $\sum \epsilon_n = \infty$ if and only if $m(S) = 0$. Cantor ternary set comes out to be a particular case of construction of Cantor like sets by choosing $\epsilon_n = \frac{1}{3}$ for all n. Similarly we can construct Cantor - $\frac{2}{5}$ set by choosing $\epsilon_n = \frac{1}{5}$ for all n. In the construction of Cantor - $\frac{2}{5}$ set the length of remaining closed intervals at each stage are equal to $(\frac{2}{5})^k$ , k = 1,2,3,………… Also we can construct Cantor - $\frac{3}{7}$ set by choosing $\epsilon_n = \frac{1}{7}$ for all n. Here in the construction of Cantor - $\frac{3}{7}$ set the length of remaining closed intervals at each stage are equal to $(\frac{3}{7})^k$ , k = 1,2,3,…………*

**KEY WORDS: -** *Cantor set, Cantor like sets.*

Date of Submission: 04-April-2015 Date of Accepted: 25-April-2015

**Lemma 1:-** Given any sequence $\{ \epsilon_n \}$ with $0< \epsilon_n <1$, $\sum \epsilon_n = \infty$ if and only if $\lim_{n\to\infty} \prod_{j=1}^{n}(1-\epsilon_j) = 0$

**Proof :-** First step: Let $\sum_{j=1}^{\infty} \epsilon_j = \infty$

Then we have to show that $\lim_{n\to\infty} \prod_{j=1}^{n}(1-\epsilon_j) = 0$ i.e. $\prod_{j=1}^{\infty}(1-\epsilon_j) = 0$

Here we use $1-\epsilon \le e^{-\epsilon}$ , $0 \le \epsilon < 1$

$\therefore \quad 1-\epsilon_i \le e^{-\epsilon_i} \qquad \forall\, i = 1,2,3,\ldots\ldots\ldots$

$$\therefore \quad 1-\epsilon_1 \le e^{-\epsilon_1}$$
$$1-\epsilon_2 \le e^{-\epsilon_2}$$
$$\vdots$$
$$\vdots$$
$$1-\epsilon_n \le e^{-\epsilon_n}$$
$$\vdots$$
$$\vdots$$

Multiplying all these inequalities we get,

$$(1-\epsilon_1)(1-\epsilon_2)\ldots\ldots\ldots(1-\epsilon_n) \le e^{-\epsilon_1}.e^{-\epsilon_2}\cdots\cdots e^{-\epsilon_n}$$

$$\prod_{j=1}^{n}(1-\epsilon_j) \le e^{-(\epsilon_1+\epsilon_2+\cdots+\epsilon_n)} \qquad \ldots\ldots\ldots\ldots(1)$$

To show that $\lim_{n\to\infty} \prod_{j=1}^{n}(1-\epsilon_j) = 0$, let $\epsilon > 0$ be given. Put M = $\log(\frac{1}{\epsilon})$.

Since $\sum \epsilon_j = \infty$ then there is N such that for n ≥ N $\Rightarrow$ $\sum_{j=1}^{n} \epsilon_j > M$

$$\sum_{j=1}^{n} \epsilon_j > \log(\tfrac{1}{\epsilon})$$

$$e^{\sum_{j=1}^{n} \epsilon_j} > \frac{1}{\epsilon}$$

$$\epsilon > \frac{1}{e^{\sum_{j=1}^{n} \epsilon_j}}$$

$$\epsilon > e^{-\sum_{j=1}^{n} \epsilon_j}$$

$$\therefore \quad e^{-\sum_{j=1}^{n} \epsilon_j} < \epsilon \qquad \text{.........................(2)}$$

From equation (1) and (2) we get ,

$$\prod_{j=1}^{n}(1-\epsilon_j) < \epsilon \text{ for all } n \geq N.$$

Thus $\lim_{n\to\infty} \prod_{j=1}^{n}(1-\epsilon_j) = 0$

$$\therefore \quad \prod_{j=1}^{\infty}(1-\epsilon_j) = 0$$

Conversely:-Let $\sum_{j=1}^{\infty} \epsilon_j < \infty$ i.e. $\sum \epsilon_j < \infty$ is convergent.

We show that $\lim_{n\to\infty} \prod_{j=1}^{n}(1-\epsilon_j) \neq 0$ .

Let $P_n = \prod_{j=1}^{n}(1-\epsilon_j)$

Since $\epsilon_j \geq 0$, $\epsilon_j \neq 1$ $\forall$ j and $\sum \epsilon_j < \infty$ .

we choose N so large that $\epsilon_N + \epsilon_{N+1} + \cdots\cdots < \frac{1}{2}$ --------------------- (3)

Then using induction we prove that for all n ≥ N, $(1-\epsilon_N)(1-\epsilon_{N+1})\cdots\cdots(1-\epsilon_n) \geq [1-(\epsilon_N + \epsilon_{N+1} + \cdots\cdots + \epsilon_n)]$

For n = N, the inequality is obvious. For n > N

If $(1-\epsilon_N)(1-\epsilon_{N+1})\cdots\cdots(1-\epsilon_n) \geq [1-(\epsilon_N + \epsilon_{N+1} + \cdots\cdots + \epsilon_n)]$ then

$(1-\epsilon_N)(1-\epsilon_{N+1})\cdots\cdots(1-\epsilon_n)(1-\epsilon_{n+1}) \geq [1-(\epsilon_N + \epsilon_{N+1} + \cdots\cdots + \epsilon_n)](1-\epsilon_{n+1})$

$= [1-(\epsilon_N + \epsilon_{N+1} + \cdots + \epsilon_{n+1})] + \epsilon_{n+1}(\epsilon_N + \epsilon_{N+1} + \cdots + \epsilon_n)$

$\geq [1-(\epsilon_N + \epsilon_{N+1} + \cdots\cdots + \epsilon_{n+1})]$

Thus $(1-\epsilon_N)(1-\epsilon_{N+1})\cdots\cdots(1-\epsilon_n)(1-\epsilon_{n+1}) \geq [1-(\epsilon_N + \epsilon_{N+1} + \cdots\cdots + \epsilon_{n+1})]$

$\therefore$ By induction the inequality holds for n ≥ N

i.e. $(1-\epsilon_N)(1-\epsilon_{N+1})\cdots\cdots(1-\epsilon_n) \geq [1-(\epsilon_N + \epsilon_{N+1} + \cdots\cdots + \epsilon_n)]$ for all n ≥ N ............(4)

Now by using equation (3) we get,

$(1-\epsilon_N)(1-\epsilon_{N+1})\cdots\cdots(1-\epsilon_n) \geq [1-\frac{1}{2}] = \frac{1}{2}$

$(1-\epsilon_N)(1-\epsilon_{N+1})\cdots\cdots(1-\epsilon_n) > \frac{1}{2}$ ,n ≥ N ........................... (5)

Now for n ≥ N, $\frac{P_n}{P_{N-1}} =$

$= (1-\epsilon_1)(1-\epsilon_2)\cdots(1-\epsilon_{N-1})(1-\epsilon_N)(1-\epsilon_{N+1})\cdots\cdots(1-\epsilon_n) \Big/ (1-\epsilon_1)(1-\epsilon_2)\cdots\cdots(1-\epsilon_{N-1})$

$= (1-\epsilon_N)(1-\epsilon_{N+1})\cdots\cdots(1-\epsilon_n) > \frac{1}{2}$ ,n ≥ N ( From equation (5))

$\therefore \quad \frac{P_n}{P_{N-1}} > \frac{1}{2} \quad , n \geq N$ ............................. (6)

$\therefore \inf_{n\geq N} \{ \frac{P_n}{P_{N-1}} \} \geq \frac{1}{2} \neq 0$ ............................. (7)

Consider

$\frac{P_n}{P_{N-1}} - \frac{P_{n+1}}{P_{N-1}} = (1-\epsilon_N)(1-\epsilon_{N+1}) \cdots\cdots (1-\epsilon_n) - (1-\epsilon_N)(1-\epsilon_{N+1}) \cdots\cdots (1-\epsilon_{n+1})$

$\frac{P_n}{P_{N-1}} - \frac{P_{n+1}}{P_{N-1}} = (1-\epsilon_N)(1-\epsilon_{N+1}) \cdots\cdots (1-\epsilon_n) \, [\, 1-(1-\epsilon_{n+1}) \,]$

$\frac{P_n}{P_{N-1}} - \frac{P_{n+1}}{P_{N-1}} = (1-\epsilon_N)(1-\epsilon_{N+1}) \cdots\cdots (1-\epsilon_n) \, \epsilon_{n+1} \geq 0$

$\therefore \quad \frac{P_n}{P_{N-1}} - \frac{P_{n+1}}{P_{N-1}} \quad \geq 0$

$\therefore$ $\{ \frac{P_n}{P_{N-1}} \}$ is monotonic decreasing and bounded below by $\frac{1}{2}$.

$\therefore \quad \mathrm{glb}_{n\geq N}\{ \frac{P_n}{P_{N-1}} \} = \lim_{n\to\infty} \frac{P_n}{P_{N-1}} \quad \geq \frac{1}{2}$

$\therefore \quad \lim_{n\to\infty} P_n \;\; \geq \frac{1}{2} \, P_{N-1}$

$\therefore \lim_{n\to\infty} P_n \; = \alpha \, P_{N-1}$, where $\alpha \geq 1/2$

$\therefore \lim_{n\to\infty} \prod_{j=1}^{n}(1-\epsilon_j) = \alpha \, \{ P_{N-1} \}$, where $\alpha \geq 1/2$

$\Rightarrow \quad \prod_{j=1}^{\infty}(1-\epsilon_j)$ is a positive number

$\therefore \quad \lim_{n\to\infty} \prod_{j=1}^{n}(1-\epsilon_j) \neq 0$

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**Lemma 2 :-**

If $0 < \epsilon_n < 1$, $n \geq 1$, $s_n = \epsilon_1 + \epsilon_2 + \cdots\cdots + \epsilon_n$ and $t_n = (1-\epsilon_1)(1-\epsilon_2)................(1-\epsilon_n)$ then $(1-s_n) \leq t_n \leq \frac{1}{(1+s_n)}$

**Proof :-** Given

$0 < \epsilon_n < 1$, $n \geq 1$, $s_n = \epsilon_1 + \epsilon_2 + \cdots\cdots + \epsilon_n$ and

$t_n = (1-\epsilon_1)(1-\epsilon_2)................(1-\epsilon_n)$

$\therefore \quad t_n \geq 1 - (\epsilon_1 + \epsilon_2 + \cdots\cdots + \epsilon_n)$ ( From equation (4) of Lemma 1)

$t_n \geq 1 - s_n$

$\therefore 1 - s_n \;\leq\; t_n$ ........................................(1)

Now,

$(1-\epsilon_1)(1+\epsilon_1) = 1 - \epsilon_1^2 < 1$

$\therefore \quad (1-\epsilon_1)(1+\epsilon_1) \quad < 1$

$(1-\epsilon_1) < \frac{1}{(1+\epsilon_1)}$

Similarly, $(1-\epsilon_2) < \frac{1}{(1+\epsilon_2)}$

$\vdots$

$\vdots$

$(1-\epsilon_n) < \frac{1}{(1+\epsilon_n)}$

Multiplying all these equations we get,

$$(1-\epsilon_1)(1-\epsilon_2)\ldots\ldots(1-\epsilon_n) \leq \frac{1}{(1+\epsilon_1)(1+\epsilon_2)\cdots(1+\epsilon_n)} \qquad \ldots\ldots(2)$$

Now

$$(1+\epsilon_1)(1+\epsilon_2)\cdots(1+\epsilon_n) = 1+(\epsilon_1+\epsilon_2+\cdots+\epsilon_n)+(\epsilon_1\epsilon_2+\epsilon_1\epsilon_3+\cdots)+(\epsilon_1\epsilon_2\epsilon_3+\cdots)+\cdots$$

$$(1+\epsilon_1)(1+\epsilon_2)\cdots(1+\epsilon_n) \geq 1+\epsilon_1+\epsilon_2+\cdots+\epsilon_n$$

$$\Rightarrow \frac{1}{(1+\epsilon_1)(1+\epsilon_2)\cdots(1+\epsilon_n)} \leq \frac{1}{1+\epsilon_1+\epsilon_2\cdots+\epsilon_n}$$

Putting in equation (2) we get,

$$(1-\epsilon_1)(1-\epsilon_2)\ldots\ldots(1-\epsilon_n) \leq \frac{1}{1+\epsilon_1+\epsilon_2\cdots+\epsilon_n}$$

$$t_n \leq \frac{1}{1+\epsilon_1+\epsilon_2\cdots+\epsilon_n}$$

$$t_n \leq \frac{1}{(1+Sn)} \qquad \ldots\ldots(3)$$

From equation (1) and (3) we get,

$$(1-s_n) \leq t_n \leq \frac{1}{(1+Sn)}$$

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**Corollary 3 :-**

If in addition $\lim s_n = \alpha$ and $\lim t_n = \beta$ then $(1-\alpha) \leq \beta \leq \frac{1}{(1+\alpha)}$ .

**Proof :-**By lemma 2 we get,

$$(1-s_n) \leq t_n \leq \frac{1}{(1+Sn)}$$

Given $\lim s_n = \alpha$ and $\lim t_n = \beta$

$$\therefore \quad (1-\alpha) \leq \beta \leq \frac{1}{(1+\alpha)}$$

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**Preposition 4 :-**

Given any sequence $\{\epsilon_n\}$ with $0< \epsilon_n <1$,there is a sequence $\{S_n\}$ of subsets of [0,1] such that $S_n \supset S_{n+1}$ ,$m(S_n) = \prod_{j=1}^{n}(1-\epsilon_j)$ and $S = \cap S_n$ is Cantor like set with

$m(S) = \prod_{n=1}^{\infty}(1-\epsilon_n)$ .

**Proof :-**

Let I = [0,1]

First stage :

We remove middle open intervals $I_{1,1}$ of length $\epsilon_1$ from [0,1]

i.e. intervals $I_{1,1} = \left(\frac{(1-\epsilon_1)}{2}, \frac{(1+\epsilon_1)}{2}\right)$.

The remaining two closed intervals are denoted by $J_{1,1} = \left[0, \frac{(1-\epsilon_1)}{2}\right]$ and $J_{1,2} = \left[\frac{(1+\epsilon_1)}{2}, 1\right]$. We get the set $S_1 = J_{1,1} \cup J_{1,2}$ with measure $(1-\epsilon_1)$ i.e. $m(S_1) = (1-\epsilon_1)$

Second stage :

Now we remove two middle open intervals $I_{1,2}, I_{2,2}$ of length $(1-\epsilon_1)\epsilon_2$ from the remaining two closed intervals $J_{1,1}$ and $J_{1,2}$ i.e. we remove $I_{1,2} = \left(\frac{(1-\epsilon_1)(1-\epsilon_2)}{4}, \frac{(1-\epsilon_1)(1+\epsilon_2)}{4}\right)$ and

$I_{2,2} = \left(\frac{4-(1-\epsilon_1)(1+\epsilon_2)}{4}, \frac{4-(1-\epsilon_1)(1-\epsilon_2)}{4}\right)$. The remaining four closed intervals are denoted by

$J_{2,1} = \left[0, \frac{(1-\epsilon_1)(1-\epsilon_2)}{4}\right]$, $J_{2,2} = \left[\frac{(1-\epsilon_1)(1+\epsilon_2)}{4}, \frac{(1-\epsilon_1)}{2}\right]$,

$J_{2,3} = \left[\frac{(1+\epsilon_1)}{2}, \frac{4-(1-\epsilon_1)(1+\epsilon_2)}{4}\right]$, $J_{2,4} = \left[\frac{4-(1-\epsilon_1)(1-\epsilon_2)}{4}, 1\right]$.

$\therefore$ Length of removed intervals = $m(I_{1,2}) + m(I_{2,2}) = (1-\epsilon_1)\epsilon_2 < (1-\epsilon_1)$

We get the set $S_2$ as union of remaining four closed intervals i.e. $S_2 = J_{2,1} \cup J_{2,2} \cup J_{2,3} \cup J_{2,4}$ with measure $(1-\epsilon_1)(1-\epsilon_2)$ i.e. $m(S_2) = (1-\epsilon_1)(1-\epsilon_2)$

$\therefore\ S_1 \supset S_2$

Third stage :

Now we remove four middle open intervals $I_{1,3}, I_{2,3}, I_{3,3}, I_{4,3}$ of length $(1-\epsilon_1)(1-\epsilon_2)\epsilon_3$ from the remaining four closed intervals $J_{2,1}, J_{2,2}, J_{2,3}$ and $J_{2,4}$ i.e. we remove

$I_{1,3} = \left(\frac{(1-\epsilon_1)(1-\epsilon_2)(1-\epsilon_3)}{8}, \frac{(1-\epsilon_1)(1-\epsilon_2)(1+\epsilon_3)}{8}\right)$,

$I_{2,3} = \left(\frac{(1-\epsilon_1)[4-(1-\epsilon_2)(1+\epsilon_3)]}{8}, \frac{(1-\epsilon_1)[4-(1-\epsilon_2)(1-\epsilon_3)]}{8}\right)$,

$I_{3,3} = \left(\frac{2(1+\epsilon_1) + 4 - (1-\epsilon_1)(1+\epsilon_2+\epsilon_3-\epsilon_2\epsilon_3)}{8}, \frac{2(1+\epsilon_1) + 4 - (1-\epsilon_1)(1+\epsilon_2-\epsilon_3+\epsilon_2\epsilon_3)}{8}\right)$,

$I_{4,3} = \left(\frac{8-(1-\epsilon_1)(1-\epsilon_2)(1+\epsilon_3)}{8}, \frac{8-(1-\epsilon_1)(1-\epsilon_2)(1-\epsilon_3)}{8}\right)$

The remaining four closed intervals are denoted by $J_{3,1} = \left[0, \frac{(1-\epsilon_1)(1-\epsilon_2)(1-\epsilon_3)}{8}\right]$,

$J_{3,2} = \left[\frac{(1-\epsilon_1)(1-\epsilon_2)(1+\epsilon_3)}{8}, \frac{(1-\epsilon_1)(1-\epsilon_2)}{4}\right]$, $J_{3,3} = \left[\frac{(1-\epsilon_1)(1+\epsilon_2)}{4}, \frac{(1-\epsilon_1)[4-(1-\epsilon_2)(1+\epsilon_3)]}{8}\right]$,

$J_{3,4} = \left[\frac{(1-\epsilon_1)[4-(1-\epsilon_2)(1-\epsilon_3)]}{8}, \frac{(1-\epsilon_1)}{2}\right]$, $J_{3,5} = \left[\frac{(1+\epsilon_1)}{2}, \frac{2(1+\epsilon_1) + 4 - (1-\epsilon_1)(1+\epsilon_2+\epsilon_3-\epsilon_2\epsilon_3)}{8}\right]$,

$J_{3,6} = \left[\frac{2(1+\epsilon_1) + 4 - (1-\epsilon_1)(1+\epsilon_2-\epsilon_3+\epsilon_2\epsilon_3)}{8}, \frac{4-(1-\epsilon_1)(1+\epsilon_2)}{4}\right]$,

$J_{3,7} = \left[\frac{4-(1-\epsilon_1)(1-\epsilon_2)}{4}, \frac{8-(1-\epsilon_1)(1-\epsilon_2)(1+\epsilon_3)}{8}\right]$, $J_{3,8} = \left[\frac{8-(1-\epsilon_1)(1-\epsilon_2)(1-\epsilon_3)}{8}, 1\right]$

Now,

Length of removed interval = m($I_{1,3}$)+m($I_{2,3}$)+m($I_{3,3}$)+m($I_{4,3}$)

$$=(1-\epsilon_1)(1-\epsilon_2)\epsilon_3 < (1-\epsilon_1)(1-\epsilon_2).$$

We get the set $S_3 = J_{3,1} \cup J_{3,2} \cup J_{3,3} \cup J_{3,4} \cup J_{3,5} \cup J_{3,6} \cup J_{3,7} \cup J_{3,8}$ with measure $(1-\epsilon_1)(1-\epsilon_2)(1-\epsilon_3)$ i.e. m($S_3$) = $(1-\epsilon_1)(1-\epsilon_2)(1-\epsilon_3)$.

Continuing in this way we can construct $S_n$, n ≥ 4 of measure $(1-\epsilon_1)(1-\epsilon_2)(1-\epsilon_3)\cdots\cdots(1-\epsilon_n)$.

$$\therefore \quad m(S_n) = (1-\epsilon_1)(1-\epsilon_2)(1-\epsilon_3)\cdots\cdots(1-\epsilon_n) = \prod_{j=1}^{n}(1-\epsilon_j)$$

Then $S_1 \supset S_2 \supset S_3 \supset S_4 \supset \cdots\cdots\cdots\cdots$.

If S = $\bigcap_{n=1}^{\infty} S_n$ then m(S) = $\lim_{n\to\infty} m(S_n)$

$$= \prod_{n=1}^{\infty}(1-\epsilon_n)$$

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**Conclusion :-** Using this Cantor like set we can construct different Cantor like sets by varying or fixing $\epsilon_n$.

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