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Steady MHD Couette Flow of An Incompressible Viscous Fluid Through A Porous Medium Between Two Infinite Parallel Plates Under Effect Of Inclined Magnetic Field

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--ABSTRACT--
In this paper, we discuss the steady hydro magnetic three dimensional couette flow of an incompressible viscous flow through a porous medium between two infinite parallel plates under the influence of effect of inclined magnetic filed, and taking viscous dissipation into account. The stationary plate is subjected to a slightly sinusoidal surface temperature with transverse sinusoidal suction velocity while moving plate is isothermal with injection velocity. The velocity and the temperature are evaluated by using perturbation technique. The expressions for the components of skin friction at the stationery plate and the insulated plate and the rate of heat transfer are also obtained and its behaviour computationally discussed with reference to the various governing parameters in detail.

KEYWORDS: *viscous fluid, steady flow, porous medium infinite parallel plates, MHD couette flow*

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I. INTRODUCTION:

Heat transfer in porous media has received considerable attention and has been the field of a number of investigations during the last two decade. The need for fundamental studies in porous media heat transfer stems from the fact that a better understanding of a host of thermal engineering applications in which porous materials are present is required. Some of examples of thermal engineering disciplines which stand to benefit from a better understanding of heat and fluid flow processes through porous media are geothermal systems, thermal insulations, grain storage, solid matrix heat exchangers, oil extraction and manufacturing numerous products in the chemical industry. The application of electromagnetic fields in controlling the heat transfer as in aerodynamic heating leads to the study of magneto hydrodynamic heat transfer. This MHD heat transfer has gained significance owing to recent advancement of space technology. The MHD heat transfer can be divided in two parts. One contains problems in which the heating is an incidental by product of electro magnetic fields as in MHD generators and pumps etc, and the second consists of problems in which the primary use of electromagnetic fields is to control the heat transfer (Toshivo Tagawa [39]). Heat transfer in channels partially filled with porous media has gained considerable attention in recent years because of its various applications in contemporary technology. These applications include porous journal bearing, blood flow in lungs or in arteries, nuclear reactors, porous flat plate collectors, packed bed thermal storage solidification of concentrated alloys, fibrous and granular insulation, grain storage and drying, paper drying, and food storage. Besides the use of porous subtracts to improve heat transfer in channels, which is considered as porous layers, finds applications in heat exchangers, electronic cooling, heat pipes, filtration and chemical reactors etc. In these applications engineers avoid filling entire channel with a solid matrix to reduce the pressure drop.

The flow between parallel plates is a classical problem that has important applications in magneto hydro dynamic power generators and pumps, accelerators, aerodynamic heating, electrostatic precipitation, polymer technology, petroleum industry, purification of crude oil and fluid droplets and spays. Hastmann and lazarus [13] studied the influence of a transverse uniform magnetic field on the flow of viscous incompressible electrically conducting fluid between two infinite parallel stationary and insulating plates. Then the problem was extended in numerous ways. Closed form solutions for the velocity fields and temperature distributions were obtained [1, 8, 36 and 38] under different physical effects. Some exact and numerical solutions for the heat transfer problem are found by Nigam (20) and Attia and Kotb (5). Attia and Kotb (5) Studied the steady MHD fully developed flow and heat transfer between two parallel plates with temperature dependent viscosity in the presence of uniform magnetic field.. Later Attia [4] extended the problem to the transient state.

The heat transfer in a viscous incompressible fluid neglecting the buoyancy force has been studied by several authors notably Alpher(1), Roming(24), Perimutten et al(21) Sielgel(25), Viskanta(40), Jain and Bansal(16), Gopalan and Sundaram(11), Hwang and Hong(15), Balaram(6), Hahn(12), Radha Krishnamacharya et al (22). But it was show by Gill and Casal (10) that viscosity variations and temperature differences on the horizontal forced convection flow between two parallel infinite plates can induce such flows as might significantly increase or decrease the tendency towards instability. Many authors have shown that the fluids with low Prandtl numbers are to a good deal more sensitive to gravitational field effects than the fluids with high Prandl number. Mehta and Shobha Sood (19) investigated the effect of temperature dependent viscosity on the heat transfer rate for a transient free convection flow along a non-isothermal vertical surface. They observed that a decrease in viscosity increases the heat transfer rate and reduces the time to reach steady state. Hazem A. Attia (14) studied the unsteady flow and heat transfer of a dusty conducting fluid between two parallel plates with variable viscosity and electric conductivity. The effect of the variation in the viscosity and electric conductivity of the fluid and the uniform magnetic field on the velocity and temperature fields for both the fluid and dust particles was discussed. O. Jumbul et al. (17) studied laminar heat transfer in parallel plates and circular ducts subject to uniform wall temperature by taking into account both viscous dissipation and fluid axial heat conductions in an infinite region. And extension of this work was done by O.Jambal et al. (18). Later M. Guria et al (12) studied the unsteady couette flow of a viscous incompressible fluid confined between parallel plates, rotating with a uniform angular velocity about an axis normal to the plates, here the flow was induced by the motion of the upper plate and the fluid and plates rotate in unison with the same angular velocity.

Channel flows of a Newtonian fluid with heat transfer have been studied by many authors [3, 4 and 8]. Gersten and Gross [9] imposed slightly sinusoidal transverse suction velocity on two dimensional flows over a plane wall and studied the resulting three dimensional flow and heat transfer. Since then, many authors have used this idea in different two dimensional flow problems. Singh et al [26, 31, 33 and 35] studied three dimensional free convection flow past an infinite vertical plate with periodic suction. The effect of viscous dissipation on three dimensional flow along a porous plate was studied by Singh [28]. Hydro magnetic effects and porous medium effects on these flows have also been studied by Singh [29, 30]. The three dimensional couette flow with transpiration cooling was studied by Singh [27], which was further extended to include MHD effects by Singh and Sharma [32, 34]. All papers quoted above, have been concerned with transverse suction velocity which causes the flow to be three dimensional. Rees [23] has considered the effect of a sinusoidal surface temperature distribution in a span wise direction, on the steady laminar free convection boundary layer flow a long the vertical surface, which also causes the flow to be three dimensional. Later Tak and Vyas [37] studied three dimensional couette flow and beat transfer in porous medium, taking viscous dissipation into account and assuming that the stationery plate exhibits a sinusoidal temperature and sinusoidal suction velocity, and the moving plate is isothermal with uniform injection velocity.

II. FORMULATION AND SOLUTION OF THE PROBLEM:

We consider the steady hydro magnetic three dimensional couette flow of an incompressible viscous fluid through a porous medium between two infinite parallel plates under the effect of inclined magnetic field with an angle of inclination β . The stationary plate is being on xz-plane and the other plate at a distance h, moving with uniform velocity U along x-direction. The stationery plate is subjected to sinusoidal surface temperature and sinusoidal suction velocity both, varying in z-direction and the moving plate is assumed to be isothermal with a uniform injection velocity and taking viscous dissipation into account. The steady hydro magnetic equations through a porous medium with inclined magnetic field are governed by the following equations.

$$\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{2.1}$$

$$v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = \upsilon\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) - \frac{\sigma\mu_e^2 H_0^2 Sin^2\beta}{\rho}u - \frac{\upsilon}{k}u \tag{2.2}$$

$$v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \upsilon\left(\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) - \frac{\sigma\mu_e^2 H_0^2 Sin^2\beta}{\rho}v - \frac{\upsilon}{k}v \tag{2.3}$$

$$v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \upsilon\left(\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) - \frac{\sigma\mu_e^2 H_0^2 Sin^2\beta}{\rho}w - \frac{\upsilon}{k}w \tag{2.4}$$

$$\rho C_p \left(v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z} \right) = K_1 \left(\frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) + \phi \tag{2.5}$$

Where ϕ is the viscous dissipation function given by

$$\phi = 2\mu \left(\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) + \mu \left\{ \left(\frac{\partial u}{\partial y} \right)^2 + \left(\frac{\partial w}{\partial y} + \frac{\partial w}{\partial z} \right)^2 + \left(\frac{\partial u}{\partial z} \right)^2 \right\},$$

u, v and w are the velocity components in the directions x, y and z respectively. ρ is the density of the fluid, p is the pressure, σ is the conductivity of the medium, μ_e is the magnetic permeability, μ is the coefficient of viscosity , k is the permeability of the medium, H_0 is the applied magnetic field, υ is the kinematic viscosity, T is the temperature, K_1 is the thermal conductivity and C_p is the specific heat of the fluid at constant pressure.

The boundary conditions are,

$$\left.\begin{array}{l} u = 0, \quad v = -V\left(1 + \psi \cos\left(\dfrac{\pi z}{h}\right)\right) \\ w = 0,\ T = T_1\left(1 + \psi \cos\left(\dfrac{\pi z}{h}\right)\right), \quad p = 0 \end{array}\right\} \quad at \quad y = 0 \tag{2.6}$$

$$\left.\begin{array}{l} u = U,\ v = -V,\ w = 0 \\ T = T_2 \quad , T_2 \succ T_1 \\ p = \dfrac{V \mu h}{k} \end{array}\right\} \quad at \quad y = h \tag{2.7}$$

Where $\psi \prec\prec 1$, U and V are constants with dimension of velocity, h and T_1 are constants with dimension of length and temperature respectively.

We introduce the non-dimensional variables are

$$z^* = \frac{z}{h},\ y^* = \frac{y}{h}, u^* = \frac{u}{U},\ v^* = \frac{v}{U},\ w^* = \frac{w}{U},\quad p^* = \frac{p}{\rho U^2},\ \theta = \frac{T - T_1}{T_2 - T_1}.$$

Using the non-dimensional variables the governing equation reduces to

$$\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{2.8}$$

$$v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = \frac{1}{R}\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) - \left(\frac{M^2 Sin^2 \beta\, D + 1}{RD} \right) u \tag{2.9}$$

$$v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{\partial p}{\partial y} + \frac{1}{R}\left(\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) - \left(\frac{M^2 Sin^2 \beta\, D + 1}{RD} \right) v \tag{2.10}$$

$$v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{\partial p}{\partial z} + \frac{1}{R}\left(\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) - \left(\frac{M^2 Sin^2 \beta\, D + 1}{RD} \right) w \tag{2.11}$$

$$v\frac{\partial \theta}{\partial y} + w\frac{\partial \theta}{\partial z} = \frac{1}{PR}\left(\frac{\partial^2 \theta}{\partial y^2} + \frac{\partial^2 \theta}{\partial z^2} \right) + \frac{E}{R}\phi \tag{2.12}$$

Where the viscous dissipation function given by

$$\phi = 2\left\{\left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right\} + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y} + \frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 \quad (2.13)$$

Corresponding boundary conditions are

$$\left.\begin{aligned} & u = 0, v = -\alpha\left(1 + \psi \cos(\pi z)\right) \\ & w = 0,\ \theta = a\psi \cos(\pi z) \\ & p = 0 \end{aligned}\right\} \quad at \quad y = 0 \qquad (2.14)$$

and

$$\left.\begin{aligned} & u = 1, v = -\alpha, \\ & w = 0,\ \theta = 1,\ p = d \end{aligned}\right\} \quad at \quad y = 1 \qquad (2.15)$$

Where the non-dimensional parameters are

$R = \dfrac{Uh}{\upsilon}$ is the Reynolds number $\quad P = \dfrac{\mu C_p}{K_1}$ is the Prandtl number

$D = \dfrac{k}{h^2}$ is the Darcy parameter $\quad M^2 = \dfrac{\sigma \mu_e^2 H_0^2 h^2}{\rho\upsilon}$ is the Hartmann number

$E = \dfrac{U^2}{C_p(T_2 - T_1)}$ is the Eckert number $\quad \alpha = \dfrac{V}{U}$ is the Suction parameter $\quad d = \dfrac{\alpha}{R\,D}$

In order to solve the above partial differential equations, we assumed to that

$$\begin{aligned} & u(y,z) = u_0(y) + \psi\, u_1(y,z) \\ & v(y,z) = v_0(y) + \psi\, v_1(y,z) \\ & w(y,z) = \psi\, w_1(y,z) \\ & p(y,z) = p_0(y) + \psi\, p_1(y,z) \\ & \theta(y,z) = \theta_0(y) + \psi\, \theta_1(y,z) \end{aligned} \qquad (2.16)$$

This is valid, since it is assumed that amplitude $\psi\ (\prec\prec 1)$ of sinusoidal suction velocity is small compare to its wave length. Substituting equation (2.16) in (2.8) to (2.15) and taking $\psi = 0$. The unperturbed quantities satisfies the following equations

$$\frac{dv_0}{dy} = 0 \qquad (2.17)$$

$$D\frac{d^2 u_0}{dy^2} + R\,D\,\alpha\frac{du_0}{dy} - (M^2 Sin^2\beta\ \ D + 1)u_0 = 0 \qquad (2.18)$$

$$\frac{dp_0}{dy} = d \qquad (2.19)$$

$$\frac{d^2\theta_0}{dy^2} + PR\,\alpha\frac{du_0}{dy} + EP\left(\frac{du_0}{dy}\right)^2 = 0 \qquad (2.20)$$

With the boundary conditions

$$\left.\begin{aligned} & u_0 = 0, v_0 = -\alpha \\ & \theta_0 = 0,\ p_0 = 0 \end{aligned}\right\} \quad at \quad y = 0 \qquad (2.21)$$

$$\left.\begin{array}{l} u_0 = 1, v_0 = -\alpha \\ \theta_0 = 1, \ p_0 = d \end{array}\right\} \quad at \quad y = 1 \tag{2.22}$$

The solutions of the equations (2.17) to (2.20) making use of the boundary conditions (2.21) and (2.22)

$$v_0(y) = -\alpha \tag{2.23}$$

$$u_0(y) = \frac{e^{m_1 y} - e^{m_2 y}}{e^{m_1} - e^{m_1}} \tag{2.24}$$

$$p_0(y) = d\ y \tag{2.25}$$

$$\theta_0(y) = C_1 + C_2 e^{-(PR\,\alpha)y} - \frac{EP}{(e^{m_1} - e^{m_1})^2}\left\{\frac{m_1 e^{2m_1 y}}{2(2m_1 + PR\,\alpha)} + \frac{m_2 e^{2m_2 y}}{2(2m_2 + PR\,\alpha)} - \right.$$

$$\left. - \frac{2m_1 m_2 e^{(m_1+m_2)y}}{(m_1+m_2)^2 + (PR\,\alpha)(m_1+m_2)}\right\} \tag{2.26}$$

Likewise the perturbed quantities in equations (2.17) to (2.20) satisfies the following equations

$$\frac{\partial v_1}{\partial y} + \frac{\partial w_1}{\partial z} = 0 \tag{2.27}$$

$$v_1 \frac{\partial u_0}{\partial y} - \alpha \frac{\partial u_1}{\partial y} = \frac{1}{R}\left(\frac{\partial^2 u_1}{\partial y^2} + \frac{\partial^2 u_1}{\partial z^2}\right) - \left(\frac{M^2 Sin^2\beta\ D + 1}{RD}\right)u_1 \tag{2.28}$$

$$-\alpha \frac{\partial v_1}{\partial y} = -\frac{\partial p_1}{\partial y} + \frac{1}{R}\left(\frac{\partial^2 v_1}{\partial y^2} + \frac{\partial^2 v_1}{\partial z^2}\right) - \left(\frac{M^2 Sin^2\beta\ D + 1}{RD}\right)v_1 \tag{2.29}$$

$$-\alpha \frac{\partial w_1}{\partial y} = -\frac{\partial p}{\partial z} + \frac{1}{R}\left(\frac{\partial^2 w_1}{\partial y^2} + \frac{\partial^2 w_1}{\partial z^2}\right) - \left(\frac{M^2 Sin^2\beta\ D + 1}{RD}\right)w_1 \tag{2.30}$$

$$-\alpha\ v \frac{\partial \theta_1}{\partial y} + v_1 \frac{\partial \theta_0}{\partial y} = \frac{1}{PR}\left(\frac{\partial^2 \theta_1}{\partial y^2} + \frac{\partial^2 \theta_1}{\partial z^2}\right) + \frac{2E}{R}\frac{\partial u_0}{\partial y}\frac{\partial u_1}{\partial y} \tag{2.31}$$

Corresponding the boundary conditions

$$\left.\begin{array}{l} u_1 = 0, v_1 = -\alpha \cos(\pi z) \\ w_1 = 0, \ \theta_1 = a\cos(\pi z) \\ p_1 = 0 \end{array}\right\} \quad at \quad y = 0 \tag{2.32}$$

and

$$\left.\begin{array}{l} u_1 = 0, \ v_1 = -\alpha \\ w_1 = 0, \ \theta_1 = 0 \\ p_1 = 0 \end{array}\right\} \quad at \quad y = 1 \tag{2.33}$$

The equations (2.27) to (2.31) are linear partial differential equations which describes perturbed three dimensional flow due to variation of the suction velocity stationery surface temperature along three directions. The form of the suction velocity stationery surface temperature suggests the following forms

$$u_1 = u_2(y)\cos(\pi z) \tag{2.34}$$

$$v_1 = v_2(y)\cos(\pi z) \tag{2.35}$$

$$w_1 = -\frac{1}{\pi} v_2'(y)\cos(\pi z) \tag{2.36}$$

$$p_1 = p_2(y)\cos(\pi z) \tag{2.37}$$

$$\theta_1 = \theta_2(y)\cos(\pi z) \tag{2.38}$$

The expressions for v_1 and w_1 have been chosen so that the equation of continuity (2.27) is trivially satisfied. The equations (2.29) and (2.30) being independent of the main flow and temperature field can be solved initially. Therefore substituting equations (2.35), (2.36) and (2.37) are in equations (2.29) and (2.30).Hence the following simultaneous ordinary differential equations are obtained.

$$D\frac{d^3 v_2}{dy^3} + RD\alpha\frac{d^2 v_2}{dy^2} - (D\pi^2 + M^2 Sin^2\beta + 1)v_2 = \pi^2 RD\, p_2 \tag{2.39}$$

$$D\frac{d^2 v_2}{dy^2} + RD\alpha\frac{dv_2}{dy} - (D\pi^2 + M^2 Sin^2\beta + 1)v_2 = RD\frac{dp_2}{dy} \tag{2.40}$$

Corresponding the boundary conditions

$$\left.\begin{aligned} &v_2 = -\alpha \quad and \\ &\frac{dv_2}{dy} = 0 \quad at \quad y = 0 \end{aligned}\right. \tag{2.41}$$

$$\left.\begin{aligned} &v_2 = 0 \quad and \\ &\frac{dv_2}{dy} = 0 \quad at \quad y = 1 \end{aligned}\right. \tag{2.42}$$

Solving the equations (2.39) and (2.4.) using the equations (2.41) and (2.42), which is straight forward and is substituted in equations (2.35) to (2.37). There fore we obtain the following expressions.

$$v_1 = \left(C_3 e^{m_3 y} + C_4 e^{m_4 y} + C_5 e^{\pi y} + C_6 e^{-\pi y}\right)\cos(\pi z) \tag{2.43}$$

$$w_1 = -\frac{1}{\pi}\left(C_3 e^{m_3 y} + C_4 e^{m_4 y} + C_5 e^{\pi y} + C_6 e^{-\pi y}\right)\cos(\pi z) \tag{2.44}$$

$$p_1 = \frac{1}{\pi^2 RD}\left(C_7 e^{m_3 y} + C_8 e^{m_4 y} + C_9 e^{\pi y} + C_{10} e^{-\pi y}\right)\cos(\pi z) \tag{2.45}$$

Now for the main flow and the temperature field, the expressions (2.34) and (2.38) are substituted in equations (2.28) and(2.31) to give the following ordinary differential equations

$$D\frac{d^2 u_2}{dy^2} + RD\alpha\frac{du_2}{dy} - (D\pi^2 + M^2 Sin^2\beta + 1)u_2 = RD\, v_2\frac{du_0}{dy} \tag{2.46}$$

$$\frac{d^2\theta_2}{dy^2} + PR\,\alpha\theta_2 - \pi^2\theta_2 = v_2 PR\frac{d\theta_0}{dy} - 2PE\frac{du_0}{dy}\frac{du_2}{dy} \tag{2.47}$$

Corresponding the boundary conditions

$$\left.\begin{aligned} &u_2 = 0 \quad and \\ &\theta_2 = 0 \quad at \quad y = 0 \end{aligned}\right. \tag{2.48}$$

and

$$u_2 = 0 \qquad and$$

$$\theta_2 = 0 \qquad at \qquad y = 1 \tag{2.49}$$

Solving equations (2.46) and (2.47) with boundary conditions (2.48) and (2.49) and substituting the solutions in the equations (2.34) and (2.35), the expressions for u_1 and θ_1 can be written as

$$u_1 = \{ C_{11} e^{m_3 y} + C_{12} e^{m_4 y} + \frac{R}{e^{m_1} - e^{m_2}} (B_1 e^{(m_1 + m_3) y} + B_2 e^{(m_1 + m_4) y} + B_3 e^{(m_1 + \pi) y} + \tag{2.50}$$

$$\theta_1 = \left\{ \left(\frac{G - e^{a_2} (F - a)}{e^{a_2} - e^{a_1}} \right) e^{a_1 y} + \left(\frac{G - e^{a_1} (F - a)}{e^{a_1} - e^{a_2}} \right) e^{a_2 y} + f(y) \right\} \cos(\pi z) \tag{2.51}$$

The skin friction components τ_x and τ_z along main flow and transverse direction and the non dimensional rate of surface heat transfer (Nusselt number) at both the plates can be calculated as

$$\tau_x = \frac{d\tau_x}{\mu U} = \left(\frac{du_0}{dy} \right)_{y=0} + \psi \left(\frac{du_2}{dy} \right)_{y=0} \cos(\pi z)$$

$$= \left(\frac{m_1 - m_2}{e^{m_1} - e^{m_2}} \right) + \psi \; \{ \; \frac{R m_5 (B - Ae^{m_6})}{(e^{m_6} - e^{m_5})(e^{m_1} - e^{m_2})} + \frac{R m_6 (B - Ae^{m_6})}{(e^{m_6} - e^{m_5})(e^{m_1} - e^{m_2})} +$$

$$+ \frac{R}{(e^{m_1} - e^{m_2})} + (\; B_1 (m_1 + m_3) + B_2 (m_1 + m_4) + B_3 (m_1 + \pi) + B_4 (m_1 - \pi) -$$

$$- B_1 (m_1 + m_3) - B_2 (m_1 + m_4) - B_3 (m_1 + \pi) - B_4 (m_1 - \pi) \;) \; \} \cos(\pi z)$$

$$\tau_z = \frac{d\tau_z}{\mu U} = \psi \left(\frac{dw_1}{dy} \right)_{y=0}$$

$$= -\frac{\psi}{\pi} \; \{ C_3 m_3^2 + C_4 m_4^2 + C_5 \pi^2 + C_6 \pi^2 \} \; \sin(\pi z)$$

Nusselt number at the stationery plate

$$Nu = -\frac{h}{T_2 - T_1} \left(\frac{\partial T}{\partial y} \right)_{y=0}$$

$$= - \left\{ - PR\,\alpha\, C_2 - \frac{EP}{(e^{m_1} - e^{m_2})^2} \left[\frac{m_1^2}{2 m_1 + PR\,\alpha} + \frac{m_2^2}{2 m_2 + PR\,\alpha} + \frac{2 m_1 m_2}{m_1 + m_2 + PR\,\alpha} \right] + \right.$$

$$\left. + \psi \left[\left(\frac{G - e^{a_2} (F - a)}{(e^{a_2} - e^{a_1})} \right) a_1 + \left(\frac{G - e^{a_1} (F - a)}{(e^{a_1} - e^{a_2})} \right) a_2 + f'(0) \right] \cos(\pi z) \right\}$$

Nusselt number at the moving plate

$$Nu = -\frac{h}{T_2 - T_1} \left(\frac{\partial T}{\partial y} \right)_{y=1}$$

$$= -\left\{ - PR\,\alpha\, C_2 e^{-PR\,\alpha} - \frac{EP}{(e^{m_1} - e^{m_2})^2}\left[\frac{m_1^2 e^{2m_1}}{2m_1 + PR\,\alpha} + \frac{m_2^2 e^{2m_2}}{2m_2 + PR\,\alpha} + \frac{2m_1 m_2 e^{m_1+m_2}}{m_1 + m_2 + PR\,\alpha}\right] + \right.$$

$$\left. + \psi \left[\left(\frac{G - e^{a_2}(F - a)}{(e^{a_2} - e^{a_1})}\right) a_1 e^{a_1} + \left(\frac{G - e^{a_1}(F - a)}{(e^{a_1} - e^{a_2})}\right) a_2 e^{a_2} + f'(1)\right] \cos(\pi z) \right\}$$

III. RESULTS AND DISCUSSION:

The computational analysis is carried out to discuss the behaviour of the velocity and temperature for different variations in the governing parameters Reynolds number R, Darcy parameter D, Hartmann number M, α the suction parameter, E Eckert number, P Prendtl number. In this chapter we discuss the steady hydro magnetic three dimensional couette flow of an incompressible viscous flow through a porous medium between two infinite parallel plates under the influence of transverse magnetic field and taking viscous dissipation into account. The stationary plate is subjected to a slightly sinusoidal surface temperature with transverse sinusoidal suction velocity while moving plate is isothermal with injection velocity. The velocity, temperature and the expressions for the components of skin friction and also the rate of heat transfer at the stationary plate and recovery factor at the insulated plate are obtained. The figures 1-4, 5-7, 8-11 represent the velocity profiles for *u*, *v* and *w* and for $z = 0,\ \varphi = 0.1$. The temperature profiles for θ displayed in the figure 12-17 for z = 0 level and for ψ = 0.1, a = 0.78, $\beta = \frac{\pi}{3}$.

At z = 0 level, the velocity components *u*, *v* and *w* enhances with increasing the Reynolds number *R*, suction parameter α and Darcy parameter *D*, while it decreases with increasing the Hartmann number *M* fixing the other parameters Fig (1-11). The temperature θ enhances with increasing the Reynolds number *R*, suction parameter α , Darcy parameter *D*, Prandtl number *P*, Eckert number *E* while it reduces with increasing the intensity of the magnetic field (Hartmann number *M*) at z = 0 level (Fig 12-17).

The absolute values of the skin friction components τ_x in plane z = 0, z = ½ and τ_z in the plane z = 0 and z = ½ are evaluated and tabulated in the tables (I-IV). The magnitude of τ_x enhances with increasing Reynolds number R, suction parameter α , Hartmann number M and Darcy parameter D for both z = 0 and z = ½ (tables I & II). Like wise the magnitude of τ_z enhances with increasing Reynolds number R, suction parameter α , and Darcy parameter D while it reduces with increasing intensity of the magnetic field M for both z = 0 and z = ½ (tables III & IV).

Tables (V-VIII) represent the variation of Nusselt number Nu for z= 0 and z = ½ and at y = 0 and 1. The rate of surface Heat transfer (Nusselt number) enhances with Reynolds number R, suction parameter α , Darcy parameter D, Prandtl number P and Eckert number E and reduces with increasing the magnetic parameter M. This can be seen that for all z and both the plates.

IV. CONCLUSIONS:

1. The velocity components *u*, *v* and *w* enhances with increasing the Reynolds number *R*, suction parameter α and Darcy parameter *D*, while it decreases with increasing the Hartmann number *M* fixing the other parameters at *z* = 0 level and $\beta = \frac{\pi}{3}$.

2. The temperature θ enhances with increasing the Reynolds number *R*, suction parameter α , Darcy parameter *D*, Prandtl number *P*, Eckert number *E* while it reduces with increasing the intensity of the magnetic field (Hartmann number *M*) at *z* = 0 level.

3. The magnitude of τ_x enhances with increasing Reynolds number R, suction parameter α , Hartmann number M and Darcy parameter D for both z = 0 and z = ½. Like wise the magnitude of τ_z enhances with increasing Reynolds number R, suction parameter α , and Darcy parameter D while it reduces with increasing intensity of the magnetic field M for both z = 0 and z = ½ at $\beta = \frac{\pi}{3}$.

5.The rate of surface Heat transfer (Nusselt number) enhances with Reynolds number *R*, suction parameter α , Darcy parameter *D*, Prandtl number *P* and Eckert number *E* and reduces with increasing the magnetic parameter *M*. This can be seen that for all z and both the plates.

GRAPHS AND TABLES

The velocity profiles in the plane z=0:

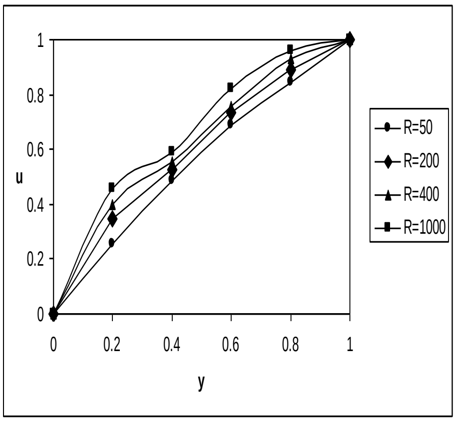


Fig. 1: The velocity component *u* with *R*
M=2, *D*=500, α =0.01, $\psi = 0.1$

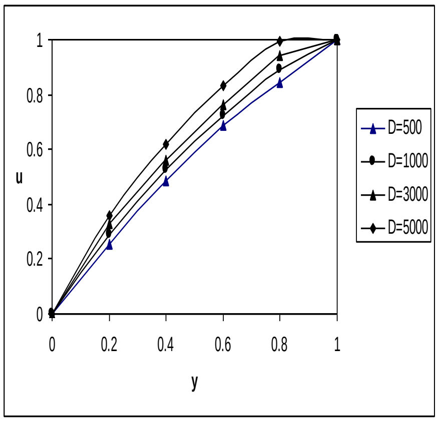


Fig. 2: The velocity component *u* with *D*
M=2, *R*=50, α =0.01, $\psi = 0.1$

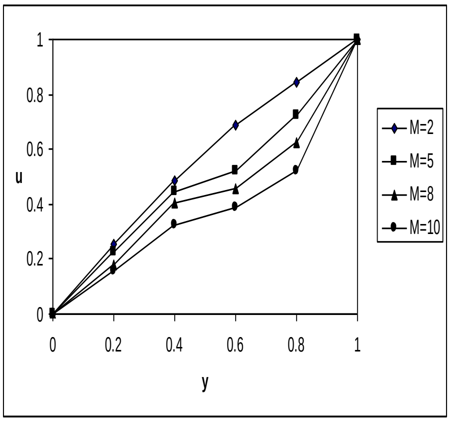


Fig. 3: The velocity component *u* with *M*
R=50, *D*=500, α =0.01, $\psi = 0.1$

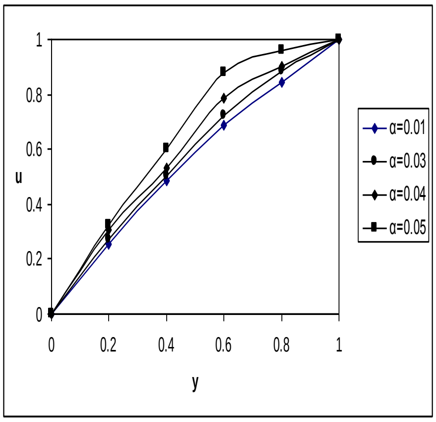


Fig. 4: The velocity component *u* with α
R=50, *D*=500, *M*=2, $\psi = 0.1$

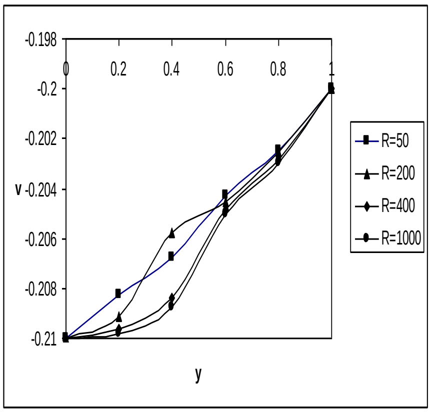


Fig. 5: The velocity component *v* with *R*
$\psi = 0.05$,$\alpha = 0.2$, *D*=500, *M*=2

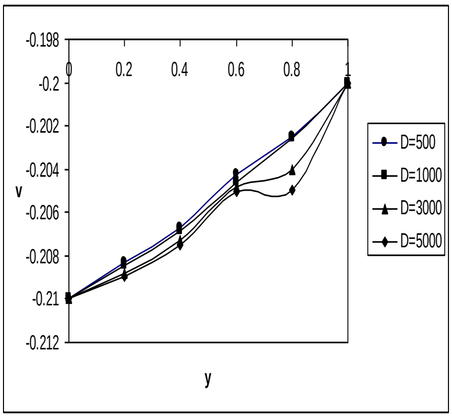


Fig. 6: The velocity component *v* with *D*
$\psi = 0.05$,$\alpha = 0.2$, *R*=50, *M*=2.

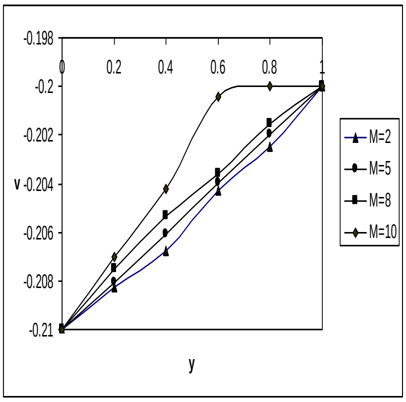


Fig. 7: The velocity component *v* with *M*
$\psi = 0.05$, $\alpha = 0.2$, *R*=50, *D*=500

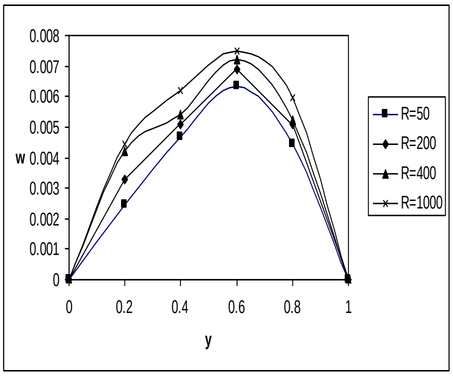


Fig. 8: The velocity component *w* with *R*
$\psi = 0.1$, *M*=2, *D*=500, α =0.01

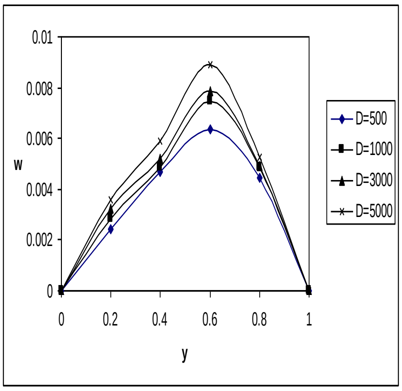
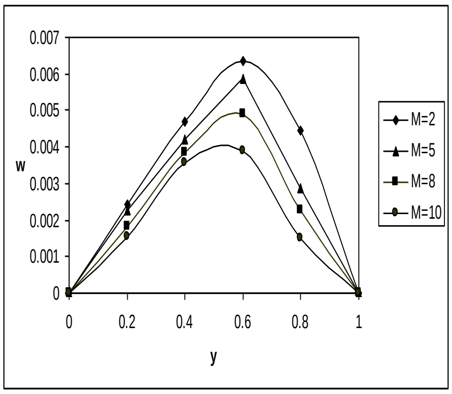


Fig. 9: The velocity component *w* with *D* .
$\psi = 0.1$, *M*=2, *R*=50, α =0.01



10: The velocity component w with M
$\psi = 0.1$, *R*=50, *D*=500, α =0.01

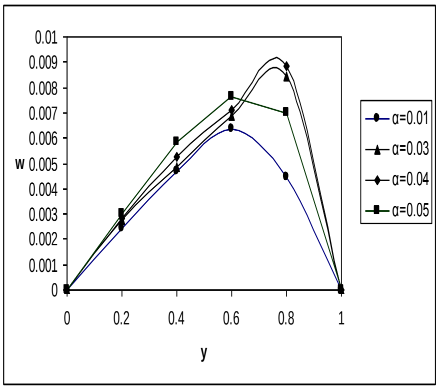


Fig 11: The velocity component *w* with α
$\psi = 0.1$, *R*=50, *D*=500, *M*=2

The temperature profiles in the plane z = 0:

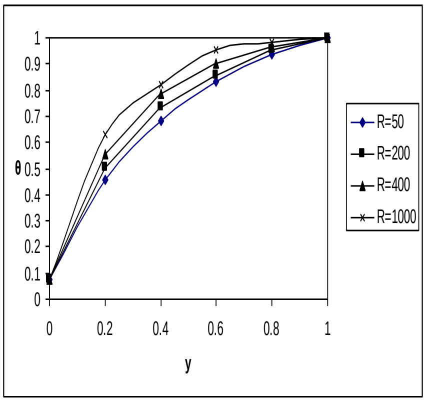


Fig.12: The temperature θ with *R* for $\psi = 0.1$
P=1.5, *E*=0.01, *M*=2, *D*=500, α =0.01, *a*=0.78

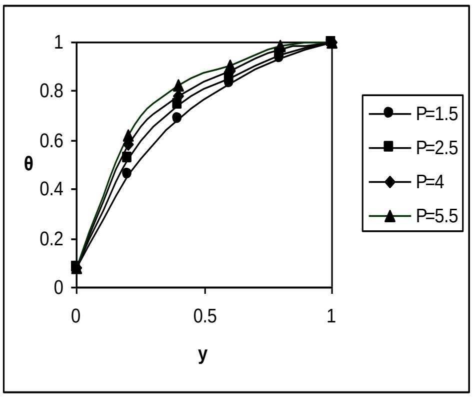


Fig.13: The temperature θ with *P* for $\psi = 0.1$
R=50, *E*=0.01, *M*=2, *D*=500, α =0.01, *a*=0.78

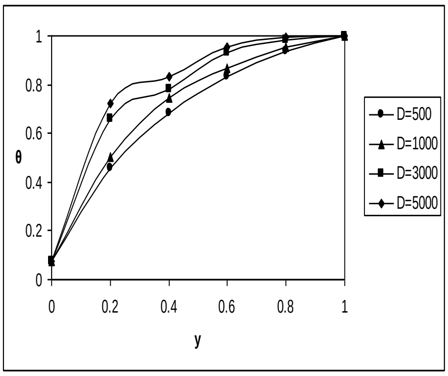


Fig.14: The temperature θ with *D* for $\psi = 0.1$
P=1.5, *E*=0.01, *M*=2, *R*=50, α =0.01, *a*=0.78

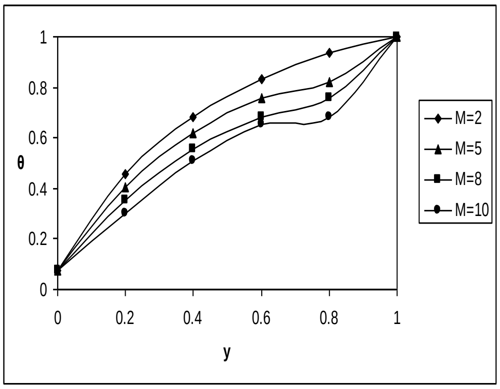


Fig.15: The temperature θ with *M* for $\psi = 0.1$
P=1.5, *E*=0.01, *R*=50, *D*=500, α =0.01, *a*=0.78

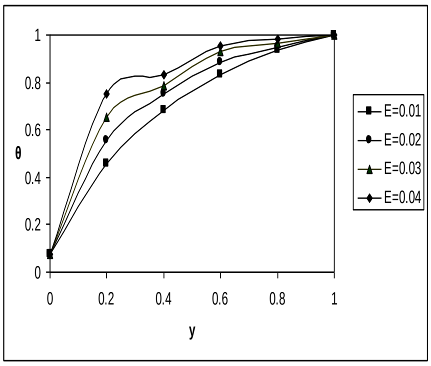


Fig.16: The temperature θ with E for $\psi = 0.1$
P=1.5, *R*=50, *M*=2, *D*=500, α =0.01, *a*=0.78

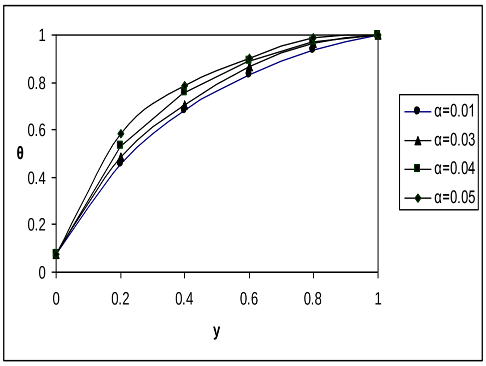


Fig.17: The temperature θ with α for $\psi = 0.1$
P=1.5, *E*=0.01, *M*=2, *D*=500, *R*=50, *a*=0.78

Table I

R	I	II	III	IV	V	VI	VII
50	0.004225	0.005821	0.006534	0.005258	0.005445	0.008241	0.012225
200	0.0062458	0.006825	0.007263	0.006314	0.006555	0.009811	0.015215
400	0.008315	0.009525	0.009845	0.008531	0.008825	0.015444	0.017225
1000	0.015615	2.018347	0.024345	0.016315	0.017315	0.024554	0.034511

	I	II	III	IV	V	VI	VII
D	500	1000	3000	500	500	500	500
α	0.01	0.01	0.01	0.03	0.05	0.01	0.01
M	2	2	2	2	2	5	8

Skin friction coefficient (τ_x) at z=0 level for $\psi = 0.1$

Table II

R	I	II	III	IV	V	VI	VII
50	0.002558	0.003185	0.003545	0.003634	0.004535	0.005245	0.006504
200	0.003125	0.005045	0.005569	0.003878	0.004838	0.006835	0.007001
400	0.004552	0.006311	0.006814	0.004875	0.005266	0.007218	0.009215
1000	0.005252	0.006341	0.007845	0.005805	0.006215	0.007404	0.014265

	I	II	III	IV	V	VI	VII
D	500	1000	3000	500	500	500	500
α	0.01	0.01	0.01	0.03	0.05	0.01	0.01
M	2	2	2	2	2	5	8

Skin friction coefficient (τ_x) at z=0.5 level for $\psi = 0.1$

Table III

R	I	II	III	IV	V	VI	VII
50	0.321854	0.384058	0.402445	0.380458	0.402145	0.281445	0.183244
200	0.566715	0.583252	0.620052	0.586585	0.626285	0.424658	0.366742
400	0.752055	0.852055	0.921455	0.788151	0.847255	0.586045	0.420022
1000	0.834155	0.943255	0.967354	0.85215	0.952698	0.690581	0.530155

	I	II	III	IV	V	VI	VII
D	500	1000	3000	500	500	500	500
α	0.01	0.01	0.01	0.03	0.05	0.01	0.01
M	2	2	2	2	2	5	8

Skin friction coefficient (τ_z) at z=0 level for $\psi = 0.1$

Table IV

R	I	II	III	IV	V	VI	VII
50	0.124458	0.243458	0.321458	0.282452	0.483652	0.094265	0.052654
200	0.310541	0.526545	0.621155	0.505151	0.580150	0.183655	0.153544
400	0.512414	0.622555	0.756841	0.798552	0.884145	0.326555	0.281555
1000	0.655488	0.712784	0.821585	0.841585	0.952144	0.420052	0.365559

	I	II	III	IV	V	VI	VII
D	500	1000	3000	500	500	500	500
α	0.01	0.01	0.01	0.03	0.05	0.01	0.01
M	2	2	2	2	2	5	8

Skin friction coefficient (τ_z) at z=0.5 level for $\psi = 0.1$

Table V

R	I	II	III	IV	V	VI	VII	VIII	IX	X	XI
50	1.245458	1.834111	2.426254	1.526444	1.826115	0.847714	0.422811	2.366458	4.224848	1.733458	1.926444
200	1.283581	1.934551	2.526555	1.814118	2.245774	0.921878	0.726144	2.578318	3.578888	2.005088	3.115585
400	1.367555	2.426458	2.843181	2.121448	2.526181	1.266118	0.926455	3.278654	4.289884	3.266884	4.726666
1000	1.431818	2.678228	3.426185	2.456888	2.826188	1.322552	1.001555	4.678265	5.078145	3.878399	5.228115

	I	II	III	IV	V	VI	VII	VIII	IX	X	XI
D	500	1000	3000	500	500	500	500	500	500	500	500
α	0.01	0.01	0.01	0.03	0.05	0.01	0.01	0.01	0.01	0.01	0.01
M	2	2	2	2	2	5	8	2	2	2	2
P	1.5	1.5	1.5	1.5	1.5	1.5	1.5	2.5	4	1.5	1.5
E	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.02	0.03

The Nusselt number (Nu) at z=0 level for $\psi = 0.1$, at y=0 (stationary plate)

Table V1

R	I	II	III	IV	V	VI	VII	VIII	IX	X	XI
50	1.875252	2.526221	2.954145	2.432445	2.878452	1.524151	1.005145	2.832651	3.842645	2.667458	3.832474
200	2.466588	2.870222	3.275015	3.005444	3.426555	2.075516	1.526158	4.275141	5.875655	3.267855	4.832647
400	2.895188	3.267555	3.857551	3.527515	4.075591	2.475455	2.005858	6.667555	7.585665	4.878485	5.926688
1000	3.266666	3.875565	4.256685	4.878515	5.783888	2.883585	2.467866	7.265252	8.875611	6.843455	7.454899

	I	II	III	IV	V	VI	VII	VIII	IX	X	XI
D	500	1000	3000	500	500	500	500	500	500	500	500
α	0.01	0.01	0.01	0.03	0.05	0.01	0.01	0.01	0.01	0.01	0.01
M	2	2	2	2	2	5	8	2	2	2	2
P	1.5	1.5	1.5	1.5	1.5	1.5	1.5	2.5	4	1.5	1.5
E	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.02	0.03

The Nusselt number (Nu) at z=0 level for $\psi = 0.1$, at y=1 (moving plate)

Table VI1

R	I	II	III	IV	V	VI	VII	VIII	IX	X	XI
50	1.524585	1.826474	2.045478	1.652615	2.152652	1.234583	1.005245	3.005251	3.632525	2.842658	3.024525
200	2.245856	2.526289	2.882658	2.526251	3.005846	1.566525	1.251515	3.526365	4.252145	3.724142	3.942525
400	2.832458	3.245555	3.952012	2.952881	3.225423	2.534151	2.025995	4.575145	4.728522	4.345256	4.632656
1000	3.266245	3.834551	4.326614	3.566484	4.338315	3.042148	2.824512	4.878655	5.245265	4.566145	6.215250

	I	II	III	IV	V	VI	VII	VIII	IX	X	XI
D	500	1000	3000	500	500	500	500	500	500	500	500
α	0.01	0.01	0.01	0.03	0.05	0.01	0.01	0.01	0.01	0.01	0.01
M	2	2	2	2	2	5	8	2	2	2	2
P	1.5	1.5	1.5	1.5	1.5	1.5	1.5	2.5	4	1.5	1.5
E	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.02	0.03

The Nusselt number (Nu) at z=0.5 level for $\psi = 0.1$, at y=0 (stationary plate)

.

Table VIII

R	I	II	III	IV	V	VI	VII	VIII	IX	X	XI
50	2.564585	2.756714	2.952614	3.834254	4.283145	2.005132	1.551457	3.562541	4.076588	4.252222	5.286588
200	3.265556	3.826652	4.275836	4.578214	5.005125	2.426163	2.069856	3.850565	5.256565	5.578355	5.856522
400	3.842647	4.275655	4.431452	5.252458	6.267652	3.581452	2.665447	4.926566	6.732525	6.931656	7.585447
1000	4.114145	4.867512	5.266132	6.050965	7.000145	3.846352	3.455566	5.866565	7.567814	7.875125	8.014412

	I	II	III	IV	V	VI	VII	VIII	IX	X	XI
D	500	1000	3000	500	500	500	500	500	500	500	500
α	0.01	0.01	0.01	0.03	0.05	0.01	0.01	0.01	0.01	0.01	0.01
M	2	2	2	2	2	5	8	2	2	2	2
P	1.5	1.5	1.5	1.5	1.5	1.5	1.5	2.5	4	1.5	1.5
E	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.02	0.03

The Nusselt number (Nu) at z=0.5 level for $\psi = 0.1$, at y=1 (moving plate)

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Appendix:

$$C_1 = \frac{1}{(e^{-PR\alpha}-1)}\left\{\frac{EP}{(e^{m_1}-e^{m_1})^2}\left\{\frac{m_1(e^{-PR\alpha}-e^{2m_1})}{2(2m_1+PR\alpha)}+\frac{m_2(e^{-PR\alpha}-e^{2m_2})}{2(2m_2+PR\alpha)}-\right.\right.$$

$$\left.\left.-\frac{2m_1m_2(e^{-PR\alpha}-e^{(m_1+m_2)})}{(m_1+m_2)^2+(PR\alpha)(m_1+m_2)}\right\}-1\right\}$$

$$C_2 = \frac{1}{(e^{-PR\alpha}-1)}\left\{\frac{EP}{(e^{m_1}-e^{m_1})^2}\left\{\frac{m_1(e^{2m_1}-1)}{2(2m_1+PR\alpha)}+\frac{m_2(e^{2m_2}-1)}{2(2m_2+PR\alpha)}-\right.\right.$$

$$\left.\left.-\frac{2m_1m_2(e^{(m_1+m_2)}-1)}{(m_1+m_2)^2+(PR\alpha)(m_1+m_2)}\right\}+1\right\}$$

$$m_1 = \frac{-R\alpha+\sqrt{(R\alpha)^2+4\left(\frac{M^2Sin^2\beta\, D+1}{D}\right)}}{2}$$

$$m_2 = \frac{-R\alpha-\sqrt{(R\alpha)^2+4\left(\frac{M^2Sin^2\beta\, D+1}{D}\right)}}{2}$$

$$C_3 = \frac{D_1}{D}, \ C_4 = \frac{D_2}{D}, \ C_5 = \frac{D_3}{D}, \ C_6 = \frac{D_4}{D}$$

$$D = \left(a_4(\pi a_{10} + \pi a_9) - a_5(m_4 a_{10} + \pi a_8) + a_6(m_4 a_9 - \pi a_8)\right) -$$

$$-\left(a_3(\pi a_{10} + \pi a_9) - a_5(m_3 a_{10} + \pi a_7) + a_6(m_3 a_9 - \pi a_7)\right) +$$

$$+\left(a_3(m_4 a_{10} + \pi a_8) - a_4(m_3 a_{10} + \pi a_7) + a_6(m_3 a_8 - m_4 a_7)\right) -$$

$$-\left(a_3(m_4 a_9 - \pi a_8) - a_4(m_3 a_9 - \pi a_7) + a_5(m_3 a_8 - m_4 a_7)\right)$$

$$D_1 = -\alpha\left(a_4(\pi a_{10} + \pi a_9) - a_5(m_4 a_{10} + \pi a_8) + a_6(m_4 a_9 - \pi a_8)\right)$$

$$D_2 = \alpha\left(a_3(\pi a_{10} + \pi a_9) - a_5(m_3 a_{10} + \pi a_7) + a_6(m_3 a_9 - \pi a_7)\right)$$

$$D_3 = -\alpha\left(a_3(m_4 a_{10} + \pi a_8) - a_4(m_3 a_{10} + \pi a_7) + a_6(m_3 a_8 - m_4 a_7)\right)$$

$$D_4 = \alpha\left(a_3(m_4 a_9 - \pi a_8) - a_4(m_3 a_9 - \pi a_7) + a_5(m_3 a_8 - m_4 a_7)\right)$$

$$a_3 = e^{m_3}, \quad a_4 = e^{m_4}, \quad a_5 = e^{\pi}, \quad a_6 = e^{-\pi},$$

$$a_7 = m_3 e^{m_3}, \ a_8 = m_4 e^{m_4}, a_9 = \pi e^{\pi}, a_{10} = -\pi e^{-\pi}$$

$$m_3 = \frac{-R\alpha + \sqrt{(R\alpha)^2 + 4\left(\dfrac{\pi^2 D + M^2 Sin^2\beta \ \ D + 1}{D}\right)}}{2}$$

$$m_4 = \frac{-R\alpha - \sqrt{(R\alpha)^2 + 4\left(\dfrac{\pi^2 D + M^2 Sin^2\beta \ \ D + 1}{D}\right)}}{2}$$

$$C_7 = D\,C_3 m_3^3 + RD\,\alpha\,C_3\,m_3^2 - (D\pi^2 + M^2 Sin^2\beta + 1)C_3 m_3$$

$$C_8 = D\,C_4 m_4^3 + RD\,\alpha\,C_4\,m_4^2 - (D\pi^2 + M^2 Sin^2\beta + 1)C_4 m_4$$

$$C_9 = D\,C_5 \pi^3 + RD\,\alpha\,C_5\,\pi^2 - (D\pi^2 + M^2 Sin^2\beta + 1)C_5 \pi$$

$$C_{10} = -D\,C_6 \pi^3 + RD\,\alpha\,C_6\,\pi^2 + (D\pi^2 + M^2 Sin^2\beta + 1)C_6 \pi$$

$$C_{11} = \frac{R(B - Ae^{m_6})}{(e^{m_6} - e^{m_5})(e^{m_1} - e^{m_2})}$$

$$C_{12} = \frac{R(B - Ae^{m_5})}{(e^{m_5} - e^{m_6})(e^{m_1} - e^{m_2})}$$

$$A = (B_1 + B_2 + B_3 + B_4) - (B_5 + B_6 + B_7 + B_8)$$

$$B = (B_1 e^{(m_1 + m_3)} + B_2 e^{(m_1 + m_4)} + B_3 e^{(m_1 + \pi)} + B_4 e^{(m_1 - \pi)}) -$$

$$- (B_5 e^{(m_2 + m_3)} + B_6 e^{(m_2 + m_4)} + B_7 e^{(m_2 + \pi)} + B_8 e^{(m_2 - \pi)})$$

$$m_5 = \frac{-R\alpha + \sqrt{(R\alpha)^2 + 4\left(\frac{\pi^2 D + M^2 Sin^2\beta D + 1}{D}\right)}}{2}$$

$$m_6 = \frac{-R\alpha + \sqrt{(R\alpha)^2 + 4\left(\frac{\pi^2 D + M^2 Sin^2\beta D + 1}{D}\right)}}{2}$$

$$B_1 = \frac{D C_3 m_1}{D(m_1 + m_3)^2 + RD\alpha(m_1 + m_3) - (\pi^2 D + M^2 Sin^2\beta D + 1)}$$

$$B_2 = \frac{D C_4 m_1}{D(m_1 + m_4)^2 + RD\alpha(m_1 + m_4) - (\pi^2 D + M^2 Sin^2\beta D + 1)}$$

$$B_3 = \frac{D C_3 m_1}{D(m_1 + \pi)^2 + RD\alpha(m_1 + \pi) - (\pi^2 D + M^2 Sin^2\beta D + 1)}$$

$$B_4 = \frac{D C_4 m_1}{D(m_1 - \pi)^2 + RD\alpha(m_1 - \pi) - (\pi^2 D + M^2 Sin^2\beta D + 1)}$$

$$B_5 = \frac{D C_3 m_2}{D(m_2 + m_3)^2 + RD\alpha(m_1 + m_3) - (\pi^2 D + M^2 Sin^2\beta D + 1)}$$

$$B_6 = \frac{D C_4 m_2}{D(m_2 + m_4)^2 + RD\alpha(m_2 + m_4) - (\pi^2 D + M^2 Sin^2\beta D + 1)}$$

$$B_7 = \frac{D C_3 m_2}{D(m_2+\pi)^2 + RD\,\alpha(m_2+\pi) - (\pi^2 D + M^2 Sin^2\beta\; D + 1)}$$

$$B_8 = \frac{D C_4 m_2}{D(m_2-\pi)^2 + RD\,\alpha(m_2-\pi) - (\pi^2 D + M^2 Sin^2\beta\; D + 1)}$$

$$a_1 = \frac{-PR\,\alpha + \sqrt{(PR\,\alpha)^2 + 4\pi^2}}{2}$$

$$a_2 = \frac{-PR\,\alpha - \sqrt{(PR\,\alpha)^2 + 4\pi^2}}{2}$$

$$F = -A_1 - A_2 - A_3 - A_4 - A_5 - A_6 - A_7 - A_8 - A_9 - A_{10} - A_{11} - A_{12} +$$

$$+ A_{13} + A_{14} + A_{15} + A_{16} - A_{17} - A_{18} + A_{19} + A_{20}$$

$$G = -A_1 e^{m_3 - PR\,\alpha} - A_2 e^{m_4 - PR\,\alpha} - A_3 e^{\pi - PR\,\alpha} - A_4 e^{-(\pi + PR\,\alpha)} - A_5 e^{m_3 + 2m_1} - A_6 e^{m_3 + 2m_2}$$

$$- A_7 e^{m_4 + 2m_1} - A_8 e^{m_4 + 2m_2} - A_9 e^{\pi + 2m_1} - A_{10} e^{\pi + 2m_2} - A_{11} e^{-\pi + 2m_1} - A_{12} e^{-\pi + 2m_2}$$

$$+ A_{13} e^{m_1 + m_2 + m_3} + A_{14} e^{m_1 + m_2 + m_4} + A_{15} e^{m_1 + m_2 + \pi} + A_{16} e^{m_1 + m_2 - \pi} - A_{17} e^{m_1 + m_5}$$

$$- A_{18} e^{m_1 + m_6} + A_{19} e^{m_1 + m_5} + A_{20} e^{m_1 + m_6}$$

$$f(y) = -A_1 e^{(m_3 - PR\,\alpha)y} - A_2 e^{(m_4 - PR\,\alpha)y} - A_3 e^{(\pi - PR\,\alpha)y} - A_4 e^{-(\pi + PR\,\alpha)y} -$$

$$- A_5 e^{(m_3 + 2m_1)y} - A_6 e^{(m_3 + 2m_2)y} - A_7 e^{(m_4 + 2m_1)y} - A_8 e^{(m_4 + 2m_2)y} - A_9 e^{(\pi + 2m_1)y} -$$

$$- A_{10} e^{(\pi + 2m_2)y} - A_{11} e^{(-\pi + 2m_1)y} - A_{12} e^{(-\pi + 2m_2)y} + A_{13} e^{(m_1 + m_2 + m_3)y} +$$

$$+ A_{14} e^{(m_1 + m_2 + m_4)y} + A_{15} e^{(m_1 + m_2 + \pi)y} + A_{16} e^{(m_1 + m_2 - \pi)y} - A_{17} e^{(m_1 + m_5)y} -$$

$$- A_{18} e^{(m_1 + m_6)y} + A_{19} e^{(m_1 + m_5)y} + A_{20} e^{(m_1 + m_6)y}$$

$$A_1 = \frac{P^2 R^2 C_3 C_2 \alpha}{(m_3 - PR\,\alpha)^2 + PR\,\alpha(m_3 - PR\,\alpha) - \pi^2}$$

$$A_2 = \frac{P^2 R^2 C_4 C_2 \alpha}{(m_4 - PR\,\alpha)^2 + PR\,\alpha(m_4 - PR\,\alpha) - \pi^2}$$

$$A_3 = \frac{P^2R^2C_5C_2\alpha}{(\pi - PR\alpha)^2 + PR\alpha(\pi - PR\alpha) - \pi^2}$$

$$A_4 = \frac{P^2R^2C_6C_2\alpha}{(\pi - PR\alpha)^2 + PR\alpha(\pi - PR\alpha) - \pi^2}$$

$$A_5 = \frac{EPRm_1}{(e^{m_1} - e^{m_2})^2}\left[\frac{PC_3m_1}{2m_1 + PR\alpha} 2B_1(m_1 + m_3)\right]\left(\frac{1}{(m_3 + 2m_1)^2 + PR\alpha(m_3 + 2m_1) - \pi^2}\right)$$

$$A_6 = \frac{EPRm_2}{(e^{m_1} - e^{m_2})^2}\left[\frac{PC_3m_2}{2m_2 + PR\alpha} 2B_5(m_2 + m_3)\right]\left(\frac{1}{(m_3 + 2m_2)^2 + PR\alpha(m_3 + 2m_2) - \pi^2}\right)$$

$$A_7 = \frac{EPRm_1}{(e^{m_1} - e^{m_2})^2}\left[\frac{PC_4m_1}{2m_1 + PR\alpha} 2B_2(m_1 + m_4)\right]\left(\frac{1}{(m_4 + 2m_1)^2 + PR\alpha(m_4 + 2m_1) - \pi^2}\right)$$

$$A_8 = \frac{EPRm_2}{(e^{m_1} - e^{m_2})^2}\left[\frac{PC_4m_2}{2m_2 + PR\alpha} 2B_6(m_2 + m_4)\right]\left(\frac{1}{(m_4 + 2m_2)^2 + PR\alpha(m_4 + 2m_2) - \pi^2}\right)$$

$$A_9 = \frac{EPRm_1}{(e^{m_1} - e^{m_2})^2}\left[\frac{PC_5m_1}{2m_1 + PR\alpha} 2B_3(m_1 + \pi)\right]\left(\frac{1}{(\pi + 2m_1)^2 + PR\alpha(\pi + 2m_1) - \pi^2}\right)$$

$$A_{10} = \frac{EPRm_2}{(e^{m_1} - e^{m_2})^2}\left[\frac{PC_5m_2}{2m_2 + PR\alpha} 2B_7(m_2 + \pi)\right]\left(\frac{1}{(\pi + 2m_2)^2 + PR\alpha(\pi + 2m_2) - \pi^2}\right)$$

$$A_{11} = \frac{EPRm_1}{(e^{m_1} - e^{m_2})^2}\left[\frac{PC_6m_1}{2m_1 + PR\alpha} 2B_4(m_1 - \pi)\right]\left(\frac{1}{(-\pi + 2m_1)^2 + PR\alpha(-\pi + 2m_1) - \pi^2}\right)$$

$$A_{12} = \frac{EPRm_2}{(e^{m_1} - e^{m_2})^2}\left[\frac{PC_6m_2}{2m_2 + PR\alpha} 2B_8(m_2 - \pi)\right]\left(\frac{1}{(-\pi + 2m_2)^2 + PR\alpha(-\pi + 2m_2) - \pi^2}\right)$$

$$A_{13} = \frac{2EPR}{(e^{m_1} - e^{m_2})^2}\left[\frac{PC_3m_1m_2}{(m_1 + m_2 + PR\alpha)} + m_1B_5(m_2 + m_3) + m_2B_1(m_1 + m_3)\right]*$$

$$* \left(\frac{1}{(m_1 + m_2 + m_3)^2 + PR\alpha(m_1 + m_2 + m_3) - \pi^2}\right)$$

$$A_{14} = \frac{2EPR}{(e^{m_1} - e^{m_2})^2}\left[\frac{PC_4 m_1 m_2}{(m_1 + m_2 + PR\alpha)} + m_1 B_6 (m_2 + m_4) + m_2 B_2 (m_1 + m_4)\right]*$$

$$*\left(\frac{1}{(m_1 + m_2 + m_4)^2 + PR\alpha(m_1 + m_2 + m_4) - \pi^2}\right)$$

$$A_{15} = \frac{2EPR}{(e^{m_1} - e^{m_2})^2}\left[\frac{PC_5 m_1 m_2}{(m_1 + m_2 + PR\alpha)} + m_1 B_7 (m_2 + \pi) + m_2 B_3 (m_1 + \pi)\right]*$$

$$*\left(\frac{1}{(m_1 + m_2 + \pi)^2 + PR\alpha(m_1 + m_2 + \pi) - \pi^2}\right)$$

$$A_{16} = \frac{2EPR}{(e^{m_1} - e^{m_2})^2}\left[\frac{PC_6 m_1 m_2}{(m_1 + m_2 + PR\alpha)} + m_1 B_8 (m_2 - \pi) + m_2 B_4 (m_1 - \pi)\right]*$$

$$*\left(\frac{1}{(m_1 + m_2 - \pi)^2 + PR\alpha(m_1 + m_2 - \pi) - \pi^2}\right)$$

$$A_{17} = \frac{2EPm_1 m_5 C_{11}}{(e^{m_1} - e^{m_2})}\left(\frac{1}{(m_1 + m_5)^2 + PR\alpha(m_1 + m_5) - \pi^2}\right)$$

$$A_{18} = \frac{2EPm_1 m_6 C_{12}}{(e^{m_1} - e^{m_2})}\left(\frac{1}{(m_1 + m_6)^2 + PR\alpha(m_1 + m_6) - \pi^2}\right)$$

$$A_{19} = \frac{2EPm_2 m_5 C_{11}}{(e^{m_1} - e^{m_2})}\left(\frac{1}{(m_1 + m_5)^2 + PR\alpha(m_1 + m_5) - \pi^2}\right)$$

$$A_{20} = \frac{2EPm_2 m_6 C_{12}}{(e^{m_1} - e^{m_2})}\left(\frac{1}{(m_1 + m_6)^2 + PR\alpha(m_1 + m_6) - \pi^2}\right)$$