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In Shooting Pool, Part II: WhereDoes The Cue Ball Go?

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ABSTRACT
This second, of a three-part paper series, describes the movement of a cue ball immediately after it is struck by the cue stick, and then, consequently, the subsequent movement of the ball. The analysis shows that immediately after the impact the ball center moves either on a normally intended straight line or on a slight parabolic curve. The curve motion is the result of either shooter error or the intentional introduction of vertical spin ("English") on the ball. Immediately after impact the ball typically slides either along the line or the curve. After sliding stops the ball rolls either along the line or on a line extension of the curve. The paper presents the geometric parameters of the lines, the curve, and the rolling initiation point. The paper quantifies how even slight shooter error produces increasingly unintended ball movement as the shot length increases.

KEYWORDS:*pocket billiards, pool shooting, rolling/sliding spheres, impact on sphere dynamics*

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I. INTRODUCTION

In Part I of this three-part series of papers we developed an algorithm for predicting the immediate post-cue-impact movement of a cue ball. But this then raises the more pressing question: What is the subsequent struck-ball movement? In this paper we attempt to provide detailed answers to this question.The illustrations of Part I show that if the cue strikes the ball along a vertical great circle, the ball center moves in a straight line, along the horizontal projection line of the cue. If, however, the ball is struck in a plane to the side of a vertical great circle plane (as in Case 3), the ball center is expected to move on a parabola as with a sliding/spinning bowling ball [1] and as documented in Reference [2]. As the energy of the sliding/spinning ball is dissipated, the contact point will come to have zero velocity, and then the ball rolls on a straight line.The title question thus evolves into three questions: 1) What are the geometric parameters of the parabola and the line?; 2) Where is the transition point?" and (most important); 3) How far will the cue ball deviate from its intended path?In providing the answers to these questions, the balance of this paper is divided into four sections with the following section listing the governing equations for the post-cue-impact ball movement. The next section then presents the solutions of the equations. Two important application examples are given in the penultimate section and the last section provides a discussion with concluding remarks.

II. GOVERNING EQUATIONS OF MOTION

Part I of this series of papers provides a development of the governing dynamical equations for the immediate post-impulse movement of a struck cue ball. The solution of these equations then defines the initial conditions for the subsequent post-impact movement. The equation modeling the post-cue-impact motion is the same as those during cue impact except for the impulsive forces. Thus from (9) to (17) of Part I we obtain the equations:

$$\mu N \cos\theta + m\, dV_X^G / dt = 0 \tag{1}$$

$$\mu N \sin\theta + m\, dV_Y^G / dt = 0 \tag{2}$$

$$N - mg = 0 \tag{3}$$

$$\mu N \sin\theta + (2/5)\, mr\, d\omega_X / dt = 0 \tag{4}$$

$$\mu N \cos\theta - (2/5)\, mr\, d\omega_Y / dt = 0 \tag{5}$$

$$d\omega_Z / dt = 0 \tag{6}$$

$$V_X^G - V^C \cos\theta - r\omega_Y = 0 \tag{7}$$

$$V_Y^G - V^C \sin\theta + r\omega_X = 0 \tag{8}$$

where, as before, v_X^G and v_Y^G are the $\mathbf{N}_X$ and $\mathbf{N}_Y$ components of the ball center, G, velocity; ω_X, ω_Y, and ω_Z are the $\mathbf{N}_X$, $\mathbf{N}_Y$, and $\mathbf{N}_Z$ components of the ball angular velocity; v^C is the magnitude of the velocity of the contact point C; θ measures the inclination of the contact point velocity v^C relative to the X-axis (or alternatively, relative to $\mathbf{N}_X$); N is the magnitude of the normal force exerted by the table on the ball; μ is the coefficient of friction between the ball and the table; m is the mass of the ball; g is the gravity acceleration; and r is the ball radius. Also, as before, X, Y, and Z are Cartesian axes with origin O at G, with the Z-axis vertical, normal to the table surface, the X-axis in the direction of the intended ball movement and the Y-axis being perpendicular to X and Z in a dextral sense. Finally, as before, $\mathbf{N}_X$, $\mathbf{N}_Y$, and $\mathbf{N}_Z$ are unit vectors parallel to X, Y, and Z.

Figures 1 and 2 illustrate the geometry where in Fig. 2 $\mathbf{n}_c$ is a unit vector parallel to $\mathbf{V}^C$ and $\mathbf{n}_{c\perp}$ is a unit vector perpendicular to $\mathbf{n}_c$ and equal to: $\mathbf{N}_Z \times \mathbf{n}_c$.

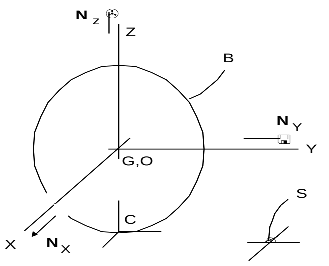

Fig. 1 Cue ball B on table surface S and point/axis/direction geometries.

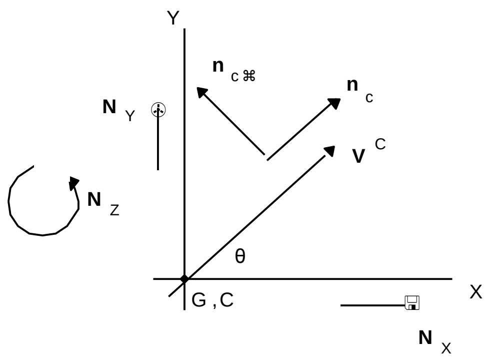

Fig. 2 Contact point velocity and unit vector geometry

In the following section we solve (1) to (8) using the immediate post-cue-impact kinematics of the ball as initial conditions.

III. SOLUTION OF THE GOVERNING EQUATIONS

From (3) and (6) we immediately see that the normal force N on the ball B, and the vertical angular velocity component of B are:

$$N = mg \quad \text{and} \quad \omega_Z = \omega_{Z0} \text{ (constant)} \tag{9}$$

Observe that the remaining six equations involve the six remaining unknowns: v_X^G, v_Y^G, v^C, ω_X, ω_Y, and θ.

Observe further that with v^C being the magnitude of $\mathbf{V}^C$, v^C can have positive values or zero but never negative. When v^C is zero, B is *rolling* on the table surface S [2] and then there is a kinematic (constraint) relation between the velocity $\mathbf{V}^G$ of G and the angular velocity $\boldsymbol{\omega}$ of B. That is

$$\mathbf{V}^G = \boldsymbol{\omega} \times r\,\mathbf{N}_Z \qquad (10)$$

or in component form:

$$V_X^G = r\omega_Y \quad \text{and} \quad V_Y^G = -r\omega_X \qquad (11)$$

Alternatively, when v^C is positive, B is *sliding* on S and then (10) and (11) are no longer valid.
For clarity in the analysis it is convenient to consider these cases separately:

Case 1.*Rolling cue ball* ($\mathbf{V}^C = 0$)
For observe that with v^C being zero (7) and (8) immediately reduce to (11).
Next, if we eliminate $\mu N \sin\theta$ between (1), (2) and (4), (5) we obtain after simplification:

$$d\omega_Y / dt = -(5/2r)dV_X^G / dt \qquad (12)$$

and

$$d\omega_X / dt = (5/2r)dV_Y^G / dt \qquad (13)$$

Then by integrating we have:

$$r\omega_Y + (5/2)V_X^G = C_1 \qquad (14)$$

and

$$-r\omega_X + (5/2)V_Y^G = C_2 \qquad (15)$$

where C_1 and C_2 are constants.
Finally, by substituting for $r\omega_Y$ and $-r\omega_X$ from (11) we have:

$$V_X^G = (2/7)C_1 \quad \text{and} \quad V_Y^G = (2/7)C_2 \qquad (16)$$

That is, V_X^G and V_Y^G are constants and consequently from (14) and (15), ω_X and ω_Y are constant. Therefore, a rolling ball rolls on a straight line at constant speed.
Case 2.*Sliding cue ball* $\left(V^C \neq 0\right)$

With the post-cue-impact contact point of the ball having non-zero velocity (ball-sliding) there will be a continuing loss of kinetic energy of the ball. Consequently, the magnitude V^C of the contact point velocity will be decreasing as the ball slides. Using (1) to (8) we can readily develop the details of the sliding ball kinematics.

To this end if we substitute for V_X^G and V_Y^G from (7) and (8) into (1) and (2) we obtain:

$$-\mu g\cos\theta - \left(dV^C / dt\right)\cos\theta + V^C\dot{\theta}\sin\theta - r\,d\omega_Y / dt = 0 \qquad (17)$$

and

$$-\mu g\sin\theta - \left(dV^C / dt\right)\sin\theta - V^C\dot{\theta}\cos\theta + r\,d\omega_X / dt = 0 \qquad (18)$$

where from (9) we have replaced the normal force N with mg.

Next, by using (4) and (5) to eliminate $d\omega_X / dt$ and $d\omega_Y / dt$, (17) and (18) become:

$$\left(dV^C / dt\right)\cos\theta - V^C\dot{\theta}\sin\theta + (7/2)\mu g\cos\theta = 0 \qquad (19)$$

and

$$\left(dV^C / dt\right)\sin\theta + V^C\dot{\theta}\cos\theta + (7/2)\mu g\cos\theta = 0 \qquad (20)$$

Finally, by solving (19) and (20) for $V^C\dot{\theta}$ and dV^C/dt we have the simple results:

$$V^C\dot{\theta} = 0 \quad \text{and} \quad dV^C/dt = -(7/2)\mu g \tag{21}$$

The first expression of (21) shows that the *direction* of the contact point velocity remains constant throughout the sliding motion of the ball. Therefore we can express θ and the unit vector $\mathbf{n}_C$ (see Fig. 2) as:

$$\theta = \theta_O \quad \text{and} \quad \mathbf{n}_C = \mathbf{n}_{CO} \tag{22}$$

where θ_O and $\mathbf{n}_{CO}$ are constants. Observe that, as a consequence, $\mathbf{n}_{C\perp}$ is also constant.
The second expression of (21) shows that the *magnitude* of the contact point velocity decreases linearly with time. That is:

$$V^C = -(7/2)\mu g t + V_O^C \tag{23}$$

Where t is measured from the immediate post-cue-impact time and V_O^C is the magnitude of the immediate post-cue-impact contact point velocity.
(23) shows that when t is $V_O^C/\mu g$ the ball stops sliding and begins rolling (in a straight line at constant speed).
With θ being constant, (1) and (2) show that the ball center acceleration components dV_X^G/dt and dV_Y^G/dt are constants. Then by using (3) the acceleration of the ball center may be expressed as:

$$\mathbf{a}^G = dV_X^G/dt\,\mathbf{N}_X + dV_Y^G/dt\,\mathbf{N}_Y = -\mu g(\cos\theta\,\mathbf{N}_X + \sin\theta\,\mathbf{N}_Y) = -\mu g\,\mathbf{n}_{CO} \tag{24}$$

(24) is seen to have the same form as the classical projectile equation [3,4]:

$$\mathbf{a}^G = -g\,\mathbf{k} \tag{25}$$

with $\mathbf{k}$ being a vertical unit vector. Therefore, as with the projectile, the ball center G moves on a parabolic path. To develop this further, observe that by integrating in (24) the velocity and position of G may be expressed as:

$$\mathbf{V}^G = -\mu g t\,\mathbf{n}_{CO} + \mathbf{V}_O^G t \tag{26}$$

and

$$\mathbf{P}^G = -\mu g t^2/2\,\mathbf{n}_{CO} + \mathbf{V}_O^G t \tag{27}$$

where $\mathbf{V}_O^G$ is the immediate post-cue-impact velocity of the ball center and where $\mathbf{P}^G$ locates G relative to its immediate post-cue-impact position.

IV. APPLICATION

Observe in (26) that if $\mathbf{V}_O^G$, the ball center velocity immediately after cue impact, is parallel to $\mathbf{n}_{CO}$ (and thus parallel to the contact point velocity), the ball moves on a line parallel to $\mathbf{n}_{OC}$. Alternatively, if $\mathbf{V}_O^G$ is not parallel to $\mathbf{n}_{OC}$, the ball will move on a parabolic curve. In this section we present simple, but yet important, practical examples illustrating both of these conditions.

Example 1. A Straight Sliding/Rolling Shot
We developed the impulse dynamics for this example in Part I, Case 2 of the paper series.

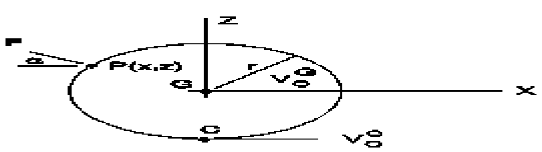

Fig. 3 Side view of the geometry of a straight shot with a downward directed cue.
Figure 3 illustrates the basic geometry of the shot where, as before, **F** is the impulsive cue force applied at point P on the vertical great circle of the ball, in the X-Z plane. Both **F** and P are in the X-Z plane. The figure also

provides a representation of the initial post-cue-impact velocities of the ball center G and the contact point C. (Note: $\mathbf{v}_O^C = \left|\mathbf{v}_O^C\right|$) Observe in Fig. 3 that the cue force is directed downward with a large inclination angle α and that C has an initial speed v_O^C along the X-axis. This means that the ball is sliding with "back-spin". That is, the ball rotation impedes the forward (X-direction) movement of the ball. Observe also that the configuration of Fig. 3 represents a typical intended shot of a pool player while attempting to avoid a "scratch" (unintended cue ball into a pocket). The downward cue angle produces back-spin, which slows the forward movement of the ball.

From Part I, Case 2, the ball is seen to move along the X-axis, (54) and (56) show v_O^G and v_O^C to be:

$$v_O^G = \left(\hat{F}/m\right)\left(\cos\alpha - \mu\sin\alpha\right) \tag{28}$$

and

$$v_O^C = \left(\hat{F}/m\right)\left\{-\left[\left(5x/2r\right)+\left(7\mu/2\right)\right]\sin\alpha + \left[1-\left(5z/2r\right)\right]\cos\alpha\right\} \tag{29}$$

where, as before, $\hat{F}$ is the impulse of **F** (that is, $\int_0^{t^*}\left|\mathbf{F}\right|dt$), m is the ball mass, and the impact time t^* is small.

Observe in (28) and (29) that both v_O^G and v_O^C are positive for practical values of α, μ, x, and z.

From (23), the contact point velocity is zero and rolling begins when t is: $2v_O^C/7\mu g$. Thus when sliding stops and rolling begins, the mass center velocity is:

$$\mathbf{V}^G = \left(v_O^G - 2v_O^C/7\right)\mathbf{n}_{CO} \tag{30}$$

Observe that if v_O^G is greater than $2v_O^C/7$ the ball will be moving forward (away from the shooter) when rolling begins – assuming it has not previously contacted an object ball.

Finally, from (27), when rolling begins the ball position is given by:

$$\mathbf{P}^G = \left(2v_O^C/7\mu g\right)\left(v_O^G - v_O^C/7\right)\mathbf{n}_{OC} \tag{31}$$

After rolling begins the ball will continue to roll on a straight line with constant speed, given in (30), until it strikes an object ball or a table rail.

Example 2. A Slight Error in Cue/Ball Point of Impact and/or Cue Direction

We also developed the impulse dynamics for this example in Part I (Case 3) of the paper series. Recall in that case a downward angled cue, intending to strike the ball at the surface intersection of the horizontal (equator) and vertical great circles, actually strikes the ball slightly (a distance η) to the left of that point (from the shooter's perspective). Observe that due to the ball symmetry, this is an error equivalent in its analysis to the cue striking the ball at the intended great circle intersection point, but with the cue erroneously angled slightly out of the vertical great circle plane. Figure 4 illustrates the geometry of the shot.

The analysis of Case 3 in Part I shows that immediately after cue impact the center and contact point velocities are [(68) and (69)]:

$$\mathbf{V}_O^G = v_{OX}^G\mathbf{N}_X + v_{OY}^G\mathbf{N}_Y \tag{32}$$

and

$$\mathbf{V}_O^C = v_O^C\mathbf{n}_{OC} \tag{33}$$

where by neglecting higher-order small terms, the components v_{OX}^G, v_{OY}^G, and v_O^C are:

$$v_{OX}^G = \left(1-\mu\alpha\right)\hat{F}/m\ ,\quad v_{OY}^G = \left(5/2\right)\left(\eta/r\right)\mu\alpha^2\left(\hat{F}/m\right)\ ,\quad \text{and}\quad v_O^C = \left\{1+\left[\left(5/2\right)-\left(7/2\right)\mu\right]\alpha\right\}\left(\hat{F}/m\right) \tag{34}$$

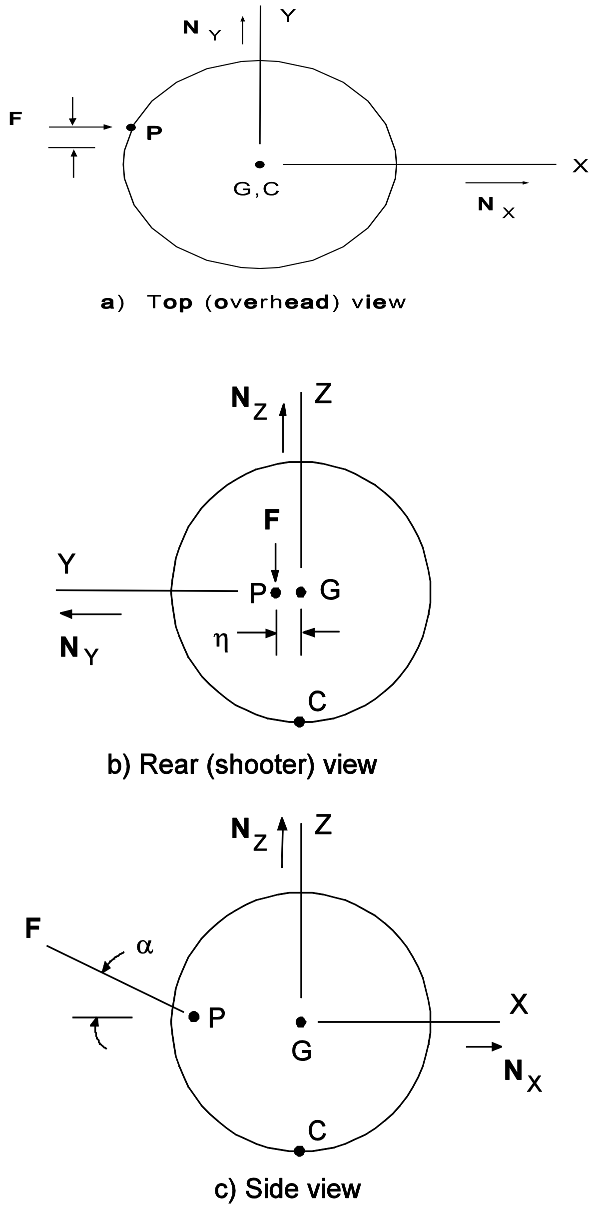

Fig 4. Cue ball struck slightly outside of the intended vertical great circle plane.

where the inclination angle θ of the contact point velocity relative to the X-axis is:

$$\theta = -5\eta\alpha / 2r \qquad (35)$$

Similarly the inclination angle φ of the immediate post-impact center velocity $\mathbf{v}_O^G$ with the X-axis is (neglecting higher-order terms):

$$\varphi = \tan\varphi = V_{OY}^G / V_{OX}^G = 5\mu\eta\alpha^2 / 2r \qquad (36)$$

Figure 5 illustrates the relative inclination of the post-cue-impact velocities of the center and contact points $\left(\mathbf{v}_O^G \text{ and } \mathbf{v}_O^C\right)$. With $\mathbf{v}_O^G$ and $\mathbf{v}_O^C$ being non-parallel and with $\mathbf{v}_O^C$ being non-zero, the ball is initially sliding and its center moves on a parabolic path. When the sliding ends, the ball will roll on a straight line at constant speed.

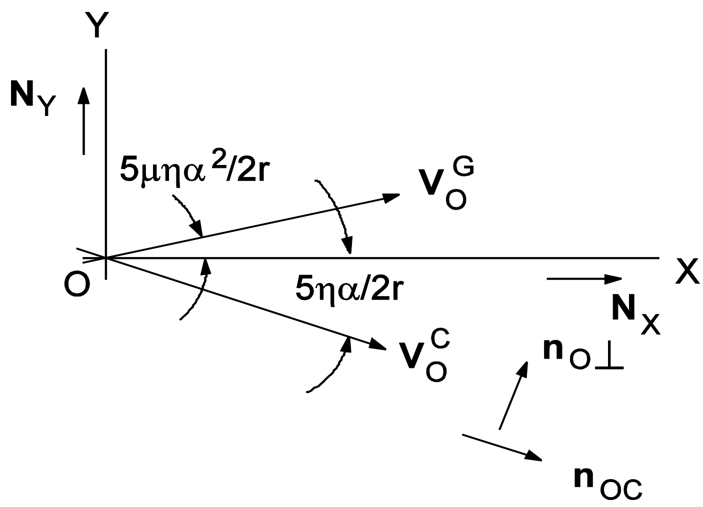

Fig. 5 Inclinations of post-cue-impact center and contact point velocities (not to scale).

These findings then raise several questions: 1) What are the coordinates of the point where sliding stops and rolling begins? 2) What is then the inclination of the rolling line? and 3) What is the subsequent rolling speed of the ball?

To answer these questions, it is convenient while the ball is sliding, to express the kinematics in terms of unit vectors $\mathbf{n}_{OC}$ and $\mathbf{n}_{C\perp}$. To this end observe from Fig. 5 that with the angle: $5\eta\alpha/2r$ being small, we have the relations:

$$\mathbf{N}_X = \mathbf{n}_{OC} + (5\eta\alpha/2r)\mathbf{n}_{C\perp} \quad \text{and} \quad \mathbf{N}_Y = -(5\eta\alpha/2r)\mathbf{n}_{OC} + \mathbf{n}_{C\perp} \tag{37}$$

Similarly, we have the inverse relations:

$$\mathbf{n}_{OC} = \mathbf{N}_X - (5\eta\alpha/2r)\mathbf{N}_Y \quad \text{and} \quad \mathbf{n}_{C\perp} = (5\eta\alpha/2r)\mathbf{N}_X + \mathbf{N}_Y \tag{38}$$

(23), (26), (34), and (37) show that while the ball is sliding, the velocities of the contact point and ball center (neglecting higher-order small terms) are:

$$\mathbf{V}^C = \left\{-(7/2)\mu g t) + \left\langle 1 + \left[(5/2) - (7/2)\mu\right]\alpha\right\rangle\left(\hat{F}/m\right)\right\}\mathbf{n}_{OC} \tag{39}$$

and

$$\mathbf{V}^G = \left[-\mu g t + \left(\hat{F}/m\right)(1-\mu\alpha)\right]\mathbf{n}_{OC} + \left(\hat{F}/m\right)(5\eta\alpha/2r)\mathbf{n}_{C\perp} \tag{40}$$

Correspondingly, (27), (34), and (37) show that while the ball is sliding, the position of the ball center G is given by:

$$\mathbf{P}^G = \left[\left(-\mu g t^2/2\right) + \left(\hat{F}/m\right)(1-\mu\alpha)t\right]\mathbf{n}_{OC} + \left(\hat{F}/m\right)(5\eta\alpha t/2r)\mathbf{n}_{C\perp} \tag{41}$$

Next, observe in (23) and (39) that the magnitude V^C of the contact point velocity decreases linearly in time. These equations show that the time t_R when V^C is zero (when sliding stops and rolling begins) is:

$$t_R = (2/7\mu g)V_O^C = \left(\hat{F}/m\mu g\right)\left[(2/7) + (5/7)\alpha - \mu\alpha\right] \tag{42}$$

By substituting the result (for t) into (40) and (41) we see that the ball center position and velocity when rolling begins are (again neglecting higher-order small terms):

$$\mathbf{P}^G = (1/\mu g)\left(\hat{F}/m\right)^2\left\{\left[(12/49) + (25/49)\alpha - \mu\alpha\right]\mathbf{n}_{OC} + (5\eta\alpha/7r)\right\}\mathbf{n}_{C\perp} \tag{43}$$

and

$$\mathbf{V}^G = \left(\hat{F}/m\right)\left[(5/7)(1-\alpha)\mathbf{n}_C + (5\eta\alpha/2r)\mathbf{n}_{C\perp}\right] \tag{44}$$

Finally, using (38) we see that the X-Y coordinates (x_R, y_R) and the $\mathbf{N}_X, \mathbf{N}_Y$ coordinates $\left(V^G_{XR}, V^G_{YR}\right)$ of the ball center position and velocity of the initiation of rolling are given by:

$$\mathbf{P}^G = x_R\mathbf{N}_X + y_R\mathbf{N}_y = (1/\mu g)\left(\hat{F}/m\right)^2\left\{\left[(12/49) + (25/49)\alpha - \mu\alpha\right]\mathbf{N}_X + (5\eta\alpha/49r)\mathbf{N}_Y\right\} \tag{45}$$

and

$$\mathbf{V}^G = V^G_{XR}\mathbf{N}_X + V^G_{YR}\mathbf{N}_Y = \left(\hat{F}/m\right)\left[(5/7)(1-\alpha)\mathbf{N}_X + (5\eta\alpha/7r)\mathbf{N}_Y\right] \tag{46}$$

(45) provides the coordinates (x_R, y_R) where rolling occurs, and (46) shows that the slope M of the rolling line and the rolling speed V^G (neglecting higher-order terms) are simply:

$$M = \eta\alpha/r \quad \text{and} \quad V^G = \left(5\hat{F}/7m\right)(1-\alpha) \tag{47}$$

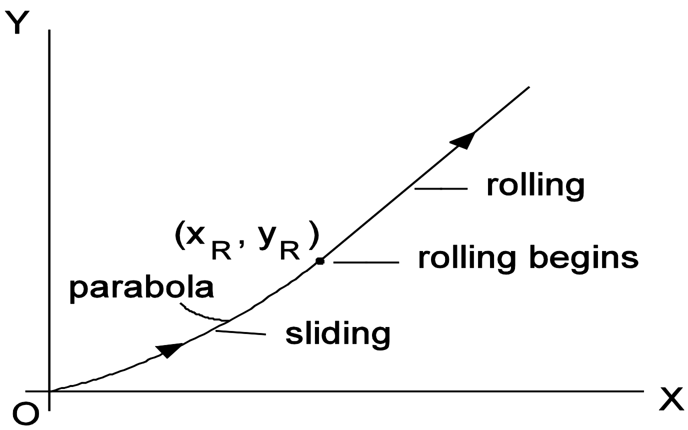

Fig. 6 Exaggerated depiction of ball center path.

Figure 6 provides an illustration (exaggerated) of the path of the ball center. When rolling begins, the ball center is: $\left[5\eta\alpha\left(\hat{F}/m\right)^2\right]/49\mu g r$ away from the X-axis – arguably a very small deviation when η and α are small. But this deviation becomes increasingly important as rolling begins and as the shot lengthens. Observe also that the shooting error, being proportional to η, may be either positive or negative, and thus the ball center path may be either to the left or to the right of the X-axis.

On rare occasions a shooter may *intentionally* strike the cue ball in a plane to the left or to the right of the vertical great circle plane of the ball [5, 6, 7]. As Mosconi [5] observed, the objective of a small out-of-vertical-plane cue impact is to introduce a vertical spin (ω_Z) on the ball ("English" or "side-spin") which will aid in positioning the cue ball after it strikes an object ball and/or a rail.

Part III presents a detailed discussion and quantification of the effects of shooter error and English.

V. DISCUSSION

These analyses and example results show that immediately after the cue impact, the ball's subsequent movement is completely established. But the characteristics of that movement are extremely sensitive to the cue's orientation and its point of impact on the ball. Indeed the ball will either go straight, as usually intended, or (more likely) the ball will move on a parabolic curve.

On most occasions there is lengthy sliding of the ball across the table due to the typical downwardly inclined cue. This cue inclination occurs due to the table rails being higher than the table surface, and due to most shooters' intent to induce back-spin on the ball to avoid a "scratch". Unless the point of impact of the cue is exactly on the vertical great circle of the ball, in the plane of the cue, the ball movement will deviate from that of a straight line. Even if this deviation is initially small, the ball position can be significantly affected, in a long shot, as it approaches an object ball. For shorter shots, however, a shooter may, on occasions, intentionally move the point of impact on the ball slightly to the left or right to obtain a favorable cue ball position after successfully striking an object ball. But this is a "trade off": A skilled shooter will accept a slight risk of an unsuccessful object ball strike in exchange for a controlled post-impact positioning of the cue ball. A skilled shooter will compensate the cue aim to account for the slight curve of the cue ball path.

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