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In Shooting Pool, Part III: Where Does the Object Ball Go? And Then Where Goes The Cue Ball?

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ABSTRACT
In this third part of the paper series, we provide answers to the most important questions in shooting pool: Will the shot be successful? And if so, how favorable will the next shot be? These are the title questions. We answer these questions by developing the post-collision kinematics of the object ball and then the cue ball. We present a generalized algorithm for simple, procedural computation of the kinematics. Effects of friction between the balls and of ball-table friction are included. We conclude with a few simple strategic principles.

***KEYWORDS**: pocket billiards, pool shooting, rolling/sliding spheres, impact on sphere dynamics*

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I. INTRODUCTION

Billiards, particularly pocket billiards (pool), is distinct amount the world's major ball games in that, with pool, multiple balls are always in play. Whereas baseball, football, basketball, soccer, tennis, golf, and bowling all use only one ball at a time, pool may involve all 16 balls (at least, in the beginning break). Nevertheless, in pool, the focus is primarily on only two balls – the cue ball and the object ball. In this third part of the paper series we use the findings and results of Parts I and II to answer the game-important title questions.
The collision of billiard balls is used in elementary texts to illustrate the conservation of momentum principle and the concept of impact/rebound modeled by a "coefficient of restitution". These elementary illustrations, however, generally neglect the effects of friction and rotational inertia. But we need to include these effects to obtain a more precise analysis of the ball dynamics, which is necessary to accurately predict the post impact ball dynamics.

During the play, there are occasions when there is no direct path between the cue and object balls. In such occasions pocketing the object ball may require a bank shot or the use of a third, or intermediate, ball. In our analysis herein we avoid these special cases and consider only those instances where there is a clear path between the cue and object balls, and also between the object ball and a pocket.The balance of the paper is divided into five sections with the first of these defining the nomenclature and geometry. Sections 3 and 4 then focus upon the dynamics of the object and cue balls, respectively. The effects of shooting errors are studied in Section 5. The final section provides a discussion and concluding remarks including implications for shooting strategies.

II. BALL-TO-BALL IMPACT GEOMETRY AND NOMENCLATURE

Imagine a typical shot attempting to "cut" an object ball toward a pocket as represented in Fig. 1, where T is the tangent plane at the contact point P between the cue and object ball. Also, in Fig. 1 α represents the "cut angle". (For a "straight shot" α is zero.) Consider next a view of the impact as in Fig. 2. As before, we introduce dextral Cartesian axes systems (for each ball) with the Z-axes being vertical and the X-axes being normal to the tangent plane as shown. The origins of the axes systems are at the ball centers G_{cue} and G_{obj} where the subscripts "cue" and "obj" are used throughout to distinguish between quantities for the cue and object balls.

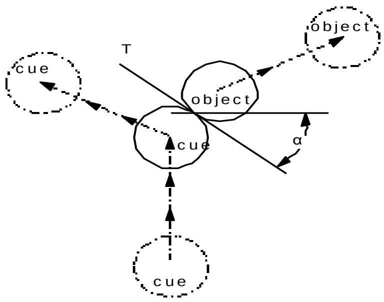

Figure 1. A Typical Cue Ball/Object Ball Shot

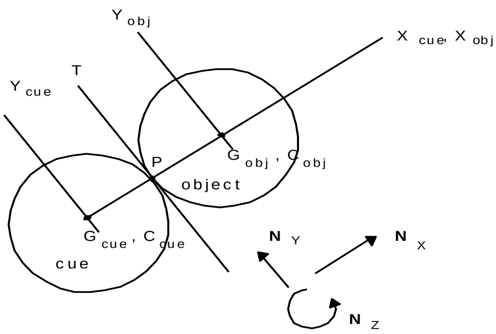

Fig. 2 Overhead View of Cue Ball/Object Ball Impact

For example, just before impact, the cue ball center velocity is expressed as:

$$\mathbf{V}^{G_{cue}} = V_{cueX}\mathbf{N}_X + V_{cueY}\mathbf{N}_Y \tag{1}$$

where $\mathbf{N}_X$, $\mathbf{N}_Y$, and $\mathbf{N}_Z$ are unit vectors parallel to the respective X, Y, and Z axes.

Similarly, the angular velocity of the cue ball just before impact is expressed as:

$$\boldsymbol{\omega}^{cue} = \omega_{cueX}\mathbf{N}_X + \omega_{cueY}\mathbf{N}_Y + \omega_{cueZ}\mathbf{N}_Z \tag{2}$$

Just after impact the cue ball center velocity and cue ball angular velocity are expressed as:

$$\overset{*}{\mathbf{V}}{}^{G_{cue}} = V^{*}_{cueX}\mathbf{N}_X + V^{*}_{cueY}\mathbf{N}_Y \tag{3}$$

and

$$\overset{*}{\omega}{}^{cue} = \omega^{*}_{cueX}\mathbf{N}_X + \omega^{*}_{cueY}\mathbf{N}_Y + \omega^{*}_{cueZ}\mathbf{N}_Z \tag{4}$$

where the * denotes immediate post impact condition.

Similarly, just after impact, the object ball center velocity and the object ball angular velocity are expressed as:

$$\overset{*}{\mathbf{V}}{}^{G_{obj}} = V^{*}_{objX}\mathbf{N}_X + V^{*}_{objY}\mathbf{N}_Y \tag{5}$$

and

$$\overset{*}{\boldsymbol{\omega}}{}^{obj} = \omega^{*}_{objX}\mathbf{N}_X + \omega^{*}_{objY}\mathbf{N}_Y + \omega^{*}_{objZ}\mathbf{N}_Z \tag{6}$$

III. OBJECT BALL DYNAMICS

While Fig. 2 shows an overhead view of the impact of the object ball by the cue ball, Fig. 3 shows a rear view, or shooter's view, of the object ball. With the point of impact on the object ball being at point P, as shown, we have impact on a horizontal great circle, which is a simpler case than the impact of a cue stick on a cue ball, as in Part I.

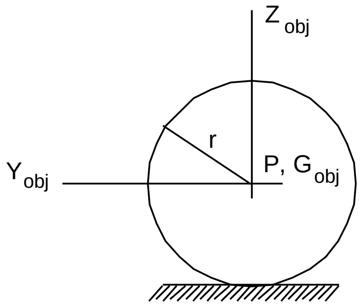

Fig. 3. Rear View (Shooter's View) of the Object Ball

In view of the analyses of Parts I and II, we can assume that we know the pre-impact kinematics of the cue ball. Then upon impact, the cue ball will exert an impulsive force, on the object ball, with magnitude F in the X-direction and corresponding impulsive friction forces in the Y and Z directions. Depending upon the impact angle α and the rotational directions of the cue ball, just before impact, the friction forces may be in either the positive or negative Y and Z directions. That is, the impulsive force $\hat{\mathbf{F}}_{obj}$ on the object ball may be expressed as:

$$\hat{\mathbf{F}}_{obj} = F\mathbf{N}_X \pm \mu_B F\mathbf{N}_Y \pm \mu_B F\mathbf{N}_Z \qquad (7)$$

where μ_B is the coefficient of friction between the balls, and where the + or – signs are determined by whether P is to the left (+) or the right (-) of the object ball center (as viewed by the shooter), and whether there is back-spin (+) or front-spin (-) of the cue ball.For competitive equipment μ_B is usually small. Marlow [1] lists μ_B as less than 0.07. Correspondingly the coefficient of friction μ between a ball and the table surface is also small typically less than 0.3 [1]. Therefore, it is reasonable to assume in the analysis that terms containing products of the friction coefficients are very small and consequently negligible.With the object ball being at rest just before impact, we can immediately use the procedures of Part I with the impulsive force of (1) to determine the post-impact movement of the object ball.

In the object ball coordinate system: $X_{obj}, Y_{obj}, Z_{obj}$ the coordinates of the impact point P are:

$$\left[x_{obj}, y_{obj}, z_{obj}\right] = \left[-r, 0, 0\right] \qquad (8)$$

where, as before, r is the ball radius.

Using the procedure in (36) to (41) of Part I, we readily obtain the following analysis procedure, or six-step algorithm:

1) Form the functions f_{objx} and f_{objy} as:

$$f_{objx} = x_{obj}\hat{F}_Z - \left[z_{obj} - (2r/5)\right]\hat{F}_X = \left[m\mu_B + (2/5)\right]rF \qquad (9)$$

and

$$f_{objy} = y_{obj}\hat{F}_Z - \left[z_{obj} - (2r/5)\right]\hat{F}_Y = \pm(2/5)\mu_B rF \qquad (10)$$

2) Form f_{obj}:

$$f_{obj} = \left(f_{objx}^2 + f_{objy}^2\right)^{1/2} = \left[(2/5)m\,\mu_B\right]rF \qquad (11)$$

3) Determine $\sin\theta_{obj}$ and $\cos\theta_{obj}$:

$$\sin\theta_{obj} = f_{objy} / f_{obj} = \pm\mu_B \text{ and } \cos\theta_{obj} = f_{objx} / f_{obj} = 1 \tag{12}$$

4) Determine θ_{obj}:

$$\theta_{obj} = \tan^{-1}\left(f_{objy} / f_{objx}\right) = \pm\mu_B \tag{13}$$

From Table 1 of Part I:

$$\begin{aligned} &\text{If } \sin\theta_{obj} = \mu_B \text{ then } 0 \le \theta_{obj} \le \pi/2 \\ &\text{If } \sin\theta_{obj} = -\mu_B \text{ then } 3\pi/2 \le \theta_{obj} \le 2\pi \end{aligned} \tag{14}$$

5) Determine V^*_{objx}, V^*_{objy}, and V^*_{objc}:

$$V^*_{objx} = \left(\hat{F}_x / m\right) + \left(\mu\hat{F}_Z / m\right)\cos\theta_{obj} = F / m \tag{15}$$

$$V^*_{objy} = \left(\hat{F}_Y / m\right) + \left(\mu\hat{F}_Z / m\right)\sin\theta_{obj} = \pm\mu_B F / m \tag{16}$$

$$V^*_{objc} = \left(5f_{obj} / 2rm\right) + \left(7\mu / 2m\right)\hat{F}_Z = \left(F / m\right)\left[1\, m\, (5/2)\mu_B\right] \tag{17}$$

6) Determine ω^*_{objX}, ω^*_{objY}, and ω^*_{objZ}:

$$\omega^*_{objX} = \left(V^*_{objc}\sin\theta_{obj} - V^*_{objY}\right)r = 0 \tag{18}$$

$$\omega^*_{objY} = \left(-V^*_{objc}\cos\theta_{obj} + V^*_{objX}\right)/ r = m\left(5\mu_B / 2r\right)\left(F / m\right) \tag{19}$$

$$\omega^*_{objZ} = \left(x_{obj}\hat{F}_Y - y_{obj}\hat{F}_X\right)\left(5 / 2mr^2\right) = m\left(5\mu_B / 2r\right)\left(F / m\right) \tag{20}$$

In obtaining these results we have neglected small terms – specifically, terms involving products of μ and μ_B. Also, V^*_{objc} denotes the immediate post-impact speed of the object ball's contact point with the table. As before, m is the ball mass. Observe the results of (11) to (18) involve the magnitude of the interactive impulsive force F. We can obtain F by considering the cue ball dynamics, as in the following section.

IV. CUE BALL DYNAMICS

In view of Figs. 1 and 2 consider an overhead view of the cue ball at the time of impact with the object ball, as in Fig. 4, where the interactive impulsive force components (F and $m\mu_B F$) from the object ball are of equal magnitude but oppositely directed from those exerted by the cue ball on the object ball.

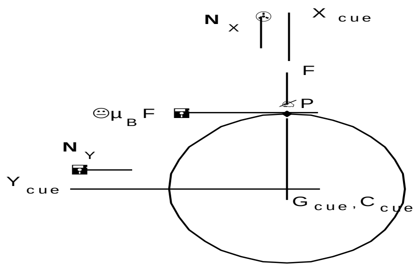

Fig. 4. Overhead View of the Cue Ball, at Impact

With the cue ball moving just prior to impact, the dynamical analysis procedure of Part I needs to be modified to account for the pre-impact movement: That is, in the integrations over the brief impact time t^* in (20) of Part I, the lower limit value is no longer zero, but instead it is the immediate pre-impact kinematic component. For example, the first expression of (20) becomes:

$$V_X^* = V_X + \int_0^{t^*} a_X \, dt \tag{21}$$

Then the integrated equilibrium equations [(21) through (26)] become:

$$\hat{F}_X - \mu \hat{N} \cos\theta_{cue} - m\left(V_{cueX}^* - V_{cueX}\right) = 0 \tag{22}$$

$$\hat{F}_Y - \mu \hat{N} \sin\theta_{cue} - m\left(V_{cueY}^* - V_{cueY}\right) = 0 \tag{23}$$

$$\hat{F}_Z - mgt^* + \hat{N} = 0 \tag{24}$$

$$y_{cue}\hat{F}_Z - z_{cue}\hat{F}_Y - \mu r\hat{N}\sin\theta_{cue} - (2/5)mr^2\left(\omega_{cueX}^* - \omega_{cueX}\right) = 0 \tag{25}$$

$$z_{cue}\hat{F}_X - x_{cue}\hat{F}_Z + \mu r\hat{N}\cos\theta_{cue} - (2/5)mr^2\left(\omega_{cueY}^* - \omega_{cueY}\right) = 0 \tag{26}$$

and

$$x_{cue}\hat{F}_Y - y_{cue}\hat{F}_X - (2/5)mr^2\left(\omega_{cueZ}^* - \omega_{cueZ}\right) = 0 \tag{27}$$

The immediate post-impact kinematic equations [(27) and (28) of Part I] remain the same and in the current format, they become:

$$V_{cueX}^* - V_{cuec}^* \cos\theta_{cue} + r\omega_{cueY}^* \tag{28}$$

and

$$V_{cueY}^* = V_{cuec}^* \sin\theta_{cue} - r\omega_{cueX}^* \tag{29}$$

(20) to (27) now form eight equations for the eight unknowns: V_{cueX}^*, V_{cueY}^*, θ_{cue}, ω_{cueX}^*, ω_{cueY}^*, ω_{cueZ}^*, V_{cuec}^*, and $\hat{N}$; and for the interactive force F, via $\hat{F}_X$, $\hat{F}_Y$, and $\hat{F}_Z$. As in Part I, the unknowns are distributed sparsely across the equations. We can thus solve the equations using Gauss elimination as before. The results immediately lead to the following six-step algorithm:

1) Form the functions f_{cueX} and f_{cueY} as:

$$f_{cueX} = y_{cue}\hat{F}_Z - \left[z_{cue} - (2r/5)\right]\hat{F}_Y + (2/5)mr\left(V_{cueY} + r\omega_{cueX}\right) \tag{30}$$

and

$$f_{cueY} = x_{cue}\hat{F}_Z - \left[z_{cue} - (2r/5)\right]\hat{F}_X + (2/5)mr\left(V_{cueX} - r\omega_{cueY}\right) \tag{31}$$

2) Form f_{cue} as:

$$f_{cue} = \left[f_{cueX}^2 + f_{cueY}^2\right]^{1/2} \tag{32}$$

3) Form $\sin\theta_{cue}$ and $\cos\theta_{cue}$:

$$\sin\theta_{cue} = f_{cueY} / f_{cue} \quad \text{and} \quad \cos\theta_{cue} = f_{cueX} / f_{cue} \tag{33}$$

4) Find θ_{cue} as:

$$\theta_{cue} = \tan^{-1}\left(f_{cueY} / f_{cueX}\right) \tag{34}$$

(The quadrant of θ_{cue} is determined from the conditions of Table I of Part I.)

5) Find Y^*_{cueX}, Y^*_{cueY}, and Y^*_{cueC} using the expressions:

$$V^*_{cueX} = V_{cueX} + \left(\hat{F}_X / m\right) + \left(\mu \hat{F}_Z / m\right)\cos\theta_{cue} \tag{35}$$

$$V^*_{cueY} = V_{cueY} + \left(\hat{F}_Y / m\right) + \left(\mu \hat{F}_Z / m\right)\sin\theta_{cue} \tag{36}$$

and

$$V_{cuec} = \left(7\mu / 2m\right)\hat{F}_Z + \left(5 f_{cue} / 2rm\right) \tag{37}$$

6) Determine ω^*_{cueX}, ω^*_{cueY}, and ω^*_{cueZ} using the expressions:

$$\omega^*_{cueX} = \left(-V^*_{cueY} / r\right) + \left(V^*_{cuec} / r\right)\sin\theta_{cue} \tag{38}$$

$$\omega^*_{cueY} = \left(V^*_{cueX} / r\right) - \left(V^*_{cuec} / r\right)\cos\theta_{cue} \tag{39}$$

and

$$\omega^*_{cueZ} = \omega_{cueZ} + \left(5 / 2mr^2\right)\left(\hat{x}\hat{F}_Y - \hat{y}\hat{F}_X\right) \tag{40}$$

Although these algorithm steps are slightly more detailed than those for a struck stationary ball, as for the cue ball of Part I, and the object ball of Sec. 3, the steps are still relatively simple, leading directly to the sought after kinematic results. The procedure is further simplified by neglecting terms with products of small quantities.

To illustrate this consider now the cue ball striking an object ball: We can obtain the impulse force components from (5) and the coordinates of the impact point by inspection from Fig. 4. That is, the impulsive force $\hat{F}_{cue}$ on the cue ball is:

$$\hat{F}_{cue} = -\hat{F}_{obj} = -F\mathbf{N}_X \; m \; \mu_B F\mathbf{N}_Y \; m \; \mu_B F\mathbf{N}_Z \tag{41}$$

From Fig. 4 the coordinates of the point of impact in cue-based axis system are:

$$\left[x_{cue}, y_{cue}, z_{cue}\right] = \left[r, 0, 0\right] \tag{42}$$

By substituting from (39) and (40) into the equations of the six-step algorithm we readily obtain the results:

$$f_{cueX} = \left[m\mu_B - (2/5)\right](rF) + (2mr/5)\left(V_{cueX} - r\omega_{cueY}\right) \tag{43}$$

$$f_{cueY} = m\left(2r\mu_B F / 5\right) + (2mr/5)\left(V_{cueY} + r\omega_{cueX}\right) \tag{44}$$

$$\theta_{cue} = \tan_{-1}\left(f_{cueY} / f_{cueX}\right) \tag{45}$$

$$V^*_{cueX} = V_{cueX} - \left(F / m\right) \tag{46}$$

$$V^*_{cueY} = V_{cueY} \; m \left(\mu_B F / m\right) \tag{47}$$

$$V^{*}_{cuec} = \left(5f\ /\ 2rm\right) \quad \text{where} \quad f = \left(f^{2}_{cueX} + f^{2}_{cueY}\right)^{1/2} \tag{48}$$

$$\omega^{*}_{cueX} = \omega_{cueX} \tag{49}$$

$$\omega^{*}_{cueY} = \left(V_{cueX}\ /\ r\right) - \left(F\ /\ mr\right) - \left(5f\ /\ 2mr^{2}\right)\cos\theta_{cue} \tag{50}$$

and

$$\omega^{*}_{cueZ} = \omega_{cueZ}\ \mathrm{m}\ \left(5\mu_{B}F\ /\ 2rm\right) \tag{51}$$

where as before we have neglected the insignificant terms.

Observe in these results that all of the terms on the right sides of the equations are expressed in terms of the cue ball kinematics just before impact and the interactive impulsive force F.

We are now in a position to determine F: From the elementary concept of impact/restitution [2,3] we have:

$$V^{*}_{objX} - V^{*}_{cueX} = eV_{cueX} \tag{52}$$

where e is the coefficient of restitution. For ball-to-ball impact with competitive equipment Marlow [1] lists e as greater than 0.9.

By substituting for V^{*}_{objX} and V^{*}_{cueX} from (13) and (44) into (50) we obtain the ratio F/m as:

$$F\ /\ m = \left[\left(1+e\right)/\ 2\right]V_{cueX} \tag{53}$$

By knowing (F/m) the immediate post-impact kinematics of both the object ball and the cue ball are known. The center velocities are:

$$V^{*}_{objX} = \left[\left(1+e\right)/\ 2\right]V_{cueX} \quad , \quad V^{*}_{objY} = \pm\mu_{B}\left[\left(1+e\right)/\ 2\right]V_{cueX} \tag{54}$$

and

$$V^{*}_{cueX} = \left[\left(1-e\right)/\ 2\right]V_{cueX} \quad , \quad V^{*}_{cueY} = V_{cueY}\ \mathrm{m}\ \mu_{B}\left[\left(1+e\right)/\ 2\right]V_{cueX} \tag{55}$$

where the upper (lower) sign is used when the cue ball strikes the left (right) side of the object ball.

V. SHOOTING ERROR ANALYSIS

In view of Figs. 1 and 2, (52) shows that with the ball-to-ball friction coefficient μ_{B} being small, the object ball moves primarily in the direction perpendicular to the contacting tangent plane T between the cue ball and the object ball. Unfortunately for the shooter the orientation of the tangent plane is extremely sensitive to the pre-impact direction of the cue ball and that this sensitivity increases with the intended “cut-angle” α (see Fig. 1) on the object ball. A small deviation in the pre-impact cue ball path has a magnified effect upon the tangent plane orientation and consequently upon the direction of the object ball movement. Moreover, this effect is amplified as the shooting distance between the cue ball and the object ball is increased. To illustrate this, consider the elementary geometric analysis of Fig. 5, where δ is the angle of deviation of the cue ball path from that of a “straight” (center-to-center) shot, and ϕ is the angle of the normal to the tangent contact plane with the center-to-center line. Thus, for small friction between the balls, ϕ is a measure of rebounding cue ball direction. The initial center-to-center distance is taken as a multiple p (not generally an integer) of the ball diameter d.

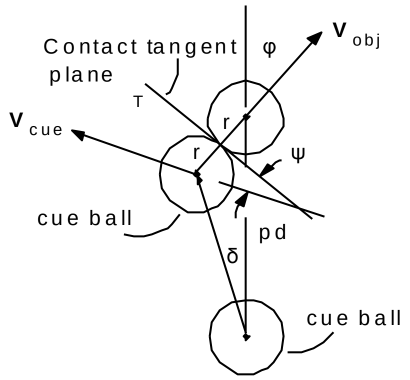

Fig. 5 Cue ball/Object ball impact geometry

If δ_{max} is the maximum value of δ for which contact occurs (that is, with right-triangle geometry), we see that the diameter multiplier p and δ_{max} are related by the simple expression:

$$\sin \delta_{max} = 1 / p \qquad (56)$$

Observe in Fig. 5 that when δ is δ_{max}, ϕ has a maximum value: ϕ_{max}, the complement of δ_{max}. Table 1 provides a listing of values of δ_{max}, ϕ_{max} and the ratio $\phi_{max} / \delta_{max}$, for initial center-to-center ball separation distances which are integer multiples of the ball diameter.

Observe in Table 1 that for large initial ball separation distances changes (errors) in pre-impact cue ball direction can produce changes 30 or more times as great in post-impact object ball direction.

Table 1. Maximum Object Ball Departure Angles as a Function of Initial Ball Separation Distance

p	δ_{max} (deg)	ϕ_{max} (deg)	$\phi_{max} / \delta_{max}$
3	19.47	70.53	3.62
4	14.48	75.52	5.21
5	11.54	78.46	6.80
7	8.21	81.78	9.96
10	5.74	84.26	14.68
15	3.82	86.18	22.54
20	2.87	87.13	30.40
30	1.91	88.09	46.11

VI. DISCUSSION

In this third part of the paper series we answer the title questions by first employing the algorithm presented in Part I for the object ball, and then for the cue ball rebound, we develop and apply a generalization of the algorithm. The result of the analysis is a simple procedure for predicting post-impact ball movement in terms of pre-impact conditions.During a game, however, a pool player cannot be expected to make trajectory calculations even with the developed simple procedures. Nevertheless, the algorithms and the accompanying error analysis establish a series of principles and strategic suggestions:

[1] Aim the cue stick so that the point of impact with the cue ball, the center of the cue ball, the desired point of contact on the object ball, and the cue stick itself, all lie in the same vertical plane.

[2] If the object ball is relatively close to a pocket, English may be used to help position the rebounding cue ball for the next shot. "English" (or Z-axis spin) can be obtained by positioning the point of impact on the cue ball, by the cue stick, slightly to the left (or to the right) of the vertical plane of the cue ball center and the desired point of impact on the object ball, while keeping the cue stick parallel to the plane. This intentional left (or right) positioning of the point of impact on the cue ball will accentuate the post-impact movement of the cue ball to the left (or right).

[3] Whenever possible, avoid long shots. Error effects increase exponentially with the distance between the cue ball and the object ball.

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