Fixed Point Theorem in Fuzzy Metric Space by Using New Implicit Relation

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Abstract
In this paper we give a fixed point theorem on fuzzy metric space with a new implicit relation. Our results extend and generalize the result of Mishra and Chaudhary [10]

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I. Introduction


II. Preliminaries

Definition 2.1 [1] A binary operation * : [0,1] × [0,1] → [0,1] is continuous t-norm if satisfies the following conditions:

(1) * is commutative and associative,
(2) * is continuous,
(3) a*1 = a for all a є [0,1],
(4) a*b ≤ c*d whenever a ≤ c and b ≤ d for all a, b, c, d є [0,1].

Examples of t-norm are a * b = min{a, b} and a * b = ab

Definition 2.3 [3] A 3-tuple (X, M, *) is said to be a fuzzy metric space if X is an arbitrary set,* is a continuous t-norm and M is a fuzzy set on X satisfying the following conditions:

The functions M(x, y, t) denote the degree of nearness between x and y with respect to t, respectively.

1) M(x, y, 0) = 0 for all x, y є X
2) M(x, y, t) = 1 for all x, y є X and t > 0 if and only if x = y
3) M(x, y, t) = M(y, x, t) for all x, y є X and t > 0
4) M(x, y, t) * M(y, z, s) ≤ M(x, z, t+s) for all x, y, z є X and s, t > 0,
5) for all x, y є X, M(x, y, .) : [0, ∞) → [0,1] is left continuous,

Remark 2.1 In a FM (X, M, *) M(x, y, .) is non-decreasing for all x, y є X.

Definition 2.4 Let (X, M, *) be a FM - space. Then
(i) A sequence {xₙ} in X is said to be a Cauchy Sequence if for all t > 0 and p > 0,
   \[ \lim_{n→∞} M(x_ₙ⁺p, xₙ, t) = 1 \]
(ii) A sequence {xₙ} in X is said to be convergent to a point x є X if for all t > 0,
   \[ \lim_{n→∞} M(xₙ, x, t) = 1 \]
Since * is continuous, the limit is uniquely determined from (5) and (11) respectively.

Definition 2.5 [11] A FM-Space (X, M, *) is said to be complete if and only if every Cauchy sequence in X is convergent.
Definition 2.6 [4] Let $A$ and $S$ be maps from a fuzzy metric space $(X, M, *)$ into itself. The maps $A$ and $S$ are said to be weakly commuting if

$$M(ASx, SAx, t) \geq M(Ax, Sz, t) \quad \text{for all } x \in X \text{ and } t > 0$$

Definition 2.7 [6] Let $A$ and $S$ be maps from an FM-space $(X, M, *)$ into itself. The maps $A$ and $S$ are said to be compatible if for all $t > 0 \lim_{n \to \infty} M(ASx_n, SAx_n, t) = 1$ whenever $(x_n)$ is a sequence in $X$ such that $\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = z$ for some $z \in X$.

Definition 2.8 [8] Two mappings $A$ and $S$ of a fuzzy metric space $(X, M, *)$ will be called reciprocally continuous if $ASu_n \to Az$, and $SAu_n \to Sz$, whenever $(u_n)$ is a sequence such that for some $u_n, Su_n \to z$ for some $z \in X$.

Definition 2.9 Let $(X, M, *)$ be a fuzzy metric space. $A$ and $S$ be self maps on $X$. A point $x$ in $X$ is called a coincidence point of $A$ and $S$ if $Ax = Sx$. In this case $w = Ax = Sx$ is called a point of coincidence of $A$ and $S$.

Definition 2.10 A pair of mappings $(A, S)$ of a fuzzy metric space $(X, M, *)$ is said to be weakly compatible if they commute at the coincident points i.e., if $Au = Su$ for some $u$ in $X$ then $ASu = SAu$.

Definition 2.11 [7] Two self maps $A$ and $S$ of a fuzzy metric space $(X, M, *)$ are said to be occasionally weakly compatible (owc) iff there is a point $x$ in $X$ which is coincidence point of $A$ and $S$ at which $A$ and $S$ commute.

Definition 2.12 (Implicit Relation) Let $\varnothing$ be the set of all real and continuous function from $(\mathbb{R}^+)^5 \to \mathbb{R}$ and such that

2.12 (i) $\varnothing$ is non increasing in 2nd, 3rd and 4th argument and

2.12 (ii) for $u, v \geq 0$, $\varnothing(u, v, v, v, v) \geq 0 \Rightarrow u \geq v$

Example $\varnothing(t_1, t_2, t_3, t_4, t_5) = t_1 - \max\{t_1, t_2, t_3, t_4\}$

Lemma 2.1 Let $(u_n)$ be a sequence in a fuzzy metric space $(X, M, *)$. If there exist a constant $k \in (0, 1)$ such that

$$M(u_{n+1}, u_{n}, t) \geq M(u_{n-1}, u_{n}, t)$$

for all $t > 0$ and $n = 1, 2, 3, ...$. Then $(u_n)$ is a Cauchy sequence in $X$.

Lemma 2.2 Let $(X, M, *)$ be a FM space and for all $x, y \in X$, $t > 0$ and if for a number $k \in (0, 1)$, $M(x, y, kt) \geq M(x, y, t)$ then $x = y$.

Lemma 2.3 [9] Let $X$ be a set, $f$ and $g$ be owc self maps of $X$. If $f$ and $g$ have a unique point of coincidence, $w = fx = gx$, then $w$ is the unique common fixed point of $f$ and $g$.

3. Main Result

Theorem 3.1 Let $(X, M, *)$ be a complete fuzzy metric space and let $A, B, S$ and $T$ be self mappings of $X$. Let the pairs $(A, S)$ and $(B, T)$ be owc. If for $\varnothing \in \varnothing$ there exist $q \in (0, 1)$ such that

$$\varnothing\left(\frac{M(Ax, By, qt), M(Sx, Ty, t), M(Sx, Ax, t), M(By, Ty, t),}{1 + M(Ax, Sx, t), M(By, Ty, t)} \cdot \frac{M(Ax, Ty, t) + M(By, Sx, t)}{2} \right) \geq 0$$

for all $x, y \in X$ and $t > 0$, then there exist a unique point $w \in X$ such that $Aw = Sw = w$ and a unique point $z \in X$ such that $Bz = Tz = z$. Moreover $z = w$, so that there is a unique common fixed point of $A, B, S$ and $T$.
**Proof**  Let the pairs \( \{A, S\} \) and \( \{B, T\} \) be owc, so there are points \( x, y \in X \) such that \( Ax = Sy \) and \( By = Ty \). We claim that \( Ax = By \). If not by inequality (1)

\[
\begin{align*}
\phi \left( \frac{M(Ax, By, qt), M(Ax, By, t), M(Sx, Ax, t), M(By, Ty, t)}{1 + M(Ax, Sx, t)} \cdot \frac{M(Ax, Ty, t) + M(By, Sx, t)}{2} \right) &\geq 0 \\
\phi \left( \frac{M(Ax, By, qt), M(Ax, By, t), M(Ax, Ax, t), M(By, By, t)}{1 + M(Ax, By, t)} \cdot \frac{M(Ax, By, t) + M(By, Ax, t)}{2} \right) &\geq 0 \\
\phi [M(Ax, By, qt), M(Ax, By, t), 1.1, M(Ax, By, t)] &\geq 0 \\
\phi [M(Ax, By, qt), M(Ax, By, t), M(Ax, By, t), M(Ax, By, t), M(Ax, By, t)] &\geq 0
\end{align*}
\]

\( \phi \) is non-increasing in 3rd and 4th argument therefore 2.12 (i) and 2.12 (ii)

\[ M(Ax, By, qt) \geq M(Ax, By, t) \]

Therefore \( Ax = By \) i.e. \( Ax = Sx = By = Ty \). Suppose that there is a unique point \( z \) such that \( Az = Sz \) then by (1) we have

\[
\begin{align*}
\phi \left( \frac{M(Az, By, qt), M(Az, By, t), M(Sz, Az, t), M(By, Ty, t)}{1 + M(Az, Sz, t)} \cdot \frac{M(Az, Ty, t) + M(By, Sz, t)}{2} \right) &\geq 0 \\
\phi [M(Az, By, qt), M(Az, By, t), 1.1, M(Az, By, t)] &\geq 0 \\
\phi [M(Az, By, qt), M(Az, By, t), M(Az, By, t), M(Az, By, t), M(Az, By, t)] &\geq 0
\end{align*}
\]

\( \phi \) is non-increasing in 3rd and 4th argument therefore 2.12 (i) and 2.12 (ii)

\[ M(Az, By, qt) \geq M(Az, By, t) \]

\( Az = By \Rightarrow Sx = Ty \). So \( Ax = Az \) and \( w = Ax = 5x \) the unique point of coincidence of \( A \) and \( S \). By lemma (2.3) \( w \) is the only common fixed point of \( A \) and \( S \). Similarly there is a unique point \( z \in X \) such that \( z = Bz = Tz \).

Assume that \( w \neq z \), we have

\[
\begin{align*}
\phi \left( \frac{M(Aw, Bz, qt), M(Aw, Bz, t), M(Sw, Aw, t), M(Bz, Tz, t)}{1 + M(Aw, Bz, t)} \cdot \frac{M(Aw, Tz, t) + M(Bz, Sw, t)}{2} \right) &\geq 0 \\
\phi \left( \frac{M(Aw, Bz, qt), M(Aw, Bz, t), M(w, z, t), M(z, z, t), M(z, z, t)}{1 + M(w, z, t)} \cdot \frac{M(w, z, t) + M(z, z, t)}{2} \right) &\geq 0 \\
\phi [M(Aw, z, qt), M(w, z, t), M(w, z, t), M(z, z, t), M(z, z, t)] &\geq 0 \\
\phi [M(Aw, z, qt), M(w, z, t), 1.1, M(w, z, t)] &\geq 0 \\
\phi [M(Aw, z, qt), M(w, z, t), M(w, z, t), M(w, z, t), M(w, z, t)] &\geq 0
\end{align*}
\]

\( \phi \) is non-increasing in 3rd and 4th argument

\[ M(Aw, Bz, qt) \geq M(w, z, t) \]
We have $z = w$ by Lemma (2.2) and $z$ is a common fixed point of $A$, $B$, $S$ and $T$. The uniqueness of the fixed point holds from (1).

**Definition 3.11 (Implicit Relation)** Let $\varphi_\theta$ be the set of all real and continuous function from $(\mathbb{R}^+)^6 \to \mathbb{R}$ and such that

3.11 (i) $\varphi_\theta$ is non increasing in $2^{nd}$, $3^{rd}$, $4^{th}$ and $5^{th}$ argument and

3.11 (ii) for $u, v \geq 0 \, \varphi(u, v, v, v, v, v) \geq 0 \Rightarrow u \geq v$ and $\varphi(u, v, v, v, v, v) \geq 0 \Rightarrow u = v$

**Theorem 3.2** Let $(X, M, *)$ be a complete fuzzy metric space and let $A$, $B$, $S$ and $T$ be self mappings of $X$. Let the pairs $\{A,S\}$ and $\{B,T\}$ be owc. If there exist $q \in (0,1)$ such that

\[
\varphi \left( \frac{M(Ax, By, qt), M(Sx, Ty, t), M(Sx, Ax, t), M(By, Ty, t),}{1 + M(By, Ty, t)} \cdot \frac{M(Ax, Ty, t) + M(By, Sx, t)}{2} \right) \geq 0
\]

\[
\varphi \left( \frac{M(Ax, By, qt), M(Ax, By, t), M(Ax, Ax, t), M(By, By, t),}{1 + M(By, Ty, t)} \cdot \frac{M(Ax, By, t) + M(By, Ax, t)}{2} \right) \geq 0
\]

\[
\varphi \left( \frac{M(Ax, By, qt), M(Ax, By, t), M(Ax, By, t), M(Ax, By, t),}{1 + M(By, Ty, t)} \cdot \frac{M(Ax, By, t) + M(By, Ax, t)}{2} \right) \geq 0
\]

\[
\varphi \left( \frac{M(Ax, By, qt), M(Ax, By, t), M(Ax, By, t), M(Ax, By, t),}{1 + M(By, Ty, t)} \cdot \frac{M(Ax, By, t) + M(By, Ax, t)}{2} \right) \geq 0
\]

$\varphi \, \varphi \, \varphi$ is non-increasing in $3^{rd}$ and $4^{th}$ argument therefore by 3.11(i) and 3.11(ii)

\[
M(Ax, By, qt) \geq M(Ax, By, t)
\]

Therefore $Ax = By$ i.e. $Ax = Sx = By = Ty$. Suppose that there is another point $z$ such that $A \cdot z = S \cdot z$ then by (2) we have $A \cdot z = S \cdot z = Ty$, So $Ax = A \cdot z$ and $w = Ax = Tz$ is the unique point of coincidence of $A$ and $T$. By lemma (2.2) $w$ is a unique point $z \in X$ such that $z = Bz = Tz$. Thus $z$ is a common fixed point of $A$, $B$, $S$ and $T$. The uniqueness of fixed point follows by (2).

**References**


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